

A mathematical solution to Yang-Mills existence and mass gap problem, one of the Millennium problems

Kimichika Fukushima

Theoretical Division, South Konandai Science Research
9-32-2-701, Konandai, Konan-ku, Yokohama 234-0054, Japan *

In this article, the Poincaré/Lorentz covariant/invariant formalism without ultraviolet divergences gives an answer to the Yang-Mills existence problem of Millennium problem (MP), since axiomatic approaches use traditional field properties. Fields are expanded using scalar plane-waves with continuous four-momentum of quantum particles. Because of asymptotic freedom, Yang-Mills theory become trivial with vanishing interactions for the infinite cutoff energy. To avoid the triviality, the main procedure introduces a cutoff. In mathematics, for the metric of the four-dimensional spacetime continuum (on which Yang-Mills fields are defined), only in the case where metric is determined from general relativity like Riemannian geometry, the cutoff scale is restricted to the Planck energy, preventing unphysical losses of intermediate states into black holes. Only for applications of mathematics to physics, self-energies due to Higgs fields will be cut off by dividing spacetime into arbitrary-shaped elements. Next, pure Yang-Mills fields composed of variational stationary classical fields and quantum fluctuations are considered. The Wilson loop for the classical field yields a linear potential between a charge and an anticharge, and the stabilization energy provides a mass gap. The action has a local mass from the classical field. These masses present an answer to the mass gap problem of the MP.

1 Introduction and Summary

This paper presents a mathematical solution to the Yang-Mills (YM) existence and mass gap problem, which is one of the Millennium problems [1]. The problem is widely known as a longstanding important problem to be solved. The aim of science is to clarify unclear problems. This paper treats solely pure Yang-Mills fields. In this paper, Sec. 2 presents a mathematical solution to the Yang-Mills (YM) existence problem, deriving Theorem 2.12. Section 3 gives a solution to the mass gap problem, leading to Theorem 3.28. Subsections 2.1 and 3.1 describe the aim and outline of Secs. 2 and 3, respectively. This study aims at constructing solid foundations of quantum field theory (QFT) for Yang-Mills fields [2]. Mathematics is a science which constructs axioms and which derives theorems from the axioms. Although mathematics is independent of physics, mathematics and physics have mutually influenced each other. In physical field theory, an expect-

*E-mail: km.fukushima@mx2.ttcn.ne.jp
Phone: +81-90-4602-0490
Phone/Fax: +81-45-831-8881

tation value for a product of operators [2–6] sometimes diverges due to high-energy contributions. This is known as the ultraviolet divergence problem, [7–15] which this paper treats. Mathematically, this expectation value with ultraviolet divergences may be in some sense expressed by a tempered distribution. This distribution is included in Schwartz distributions, and its Fourier transforms are available. On the other hand, non-Abelian Yang-Mills fields should physically gain a mass, which leads to a transition from long-range interactions to short-range interactions. A set of these mathematical requirements is the Yang-Mills existence and mass gap problem. A physical formulation of the themes including a solution to the above problems has been presented in previous papers [16–20]. Other approaches in different motivations are seen in the literatures [21–23].

On the basis of this physical formulation, this paper aims at giving a mathematical solution to the Yang-Mills existence and mass gap problem. Tomonaga (S. Tomonaga) stated that when the problem is difficult, one way is to learn theoretical schemes from nature. For this reason, two theories with associated lemmas in this paper are constructed mathematically, referring to physical meanings. (If a reference to experimental input is expressed by an axiom in mathematics, this axiom is not excluded in mathematics.) First, in Sec. 2, a mathematically well-defined field theory is formulated, presenting Theorem 2.12. Fields are expanded in terms of scalar expansion functions, which are plane waves as a function of an inner product between a four-vector in the four-dimensional (4D) spacetime continuum and a four-momentum. As mentioned below in 2.4 in the main procedure of this paper, Yang-Mills fields have a specific feature of asymptotic freedom. Interactions disappear and particles are free for the infinite cutoff energy. Namely, the Yang-Mills theory is trivial for the infinite high energy. In order to avoid this triviality, this paper introduces a finite cutoff in the main procedure for Yang-Mills fields. (The cutoff scale is not specified at this stage.) Quantities, corresponding physically to vacuum expectation values for a product of operators, are formulated using this cutoff. As described below in Subsec. 2.2, in mathematics like for Riemannian geometry, it does not necessarily seem that an axiom for the metric of the spacetime continuum should not be determined from general relativity. Only if general relativity is considered, the metric gives rise to black holes at the Planck energy (fundamental unit energy of about 10^{28} eV (10^{19} GeV)), missing particles probabilities unphysically in high-energy intermediate stats. Only when the metric is induced from general relativity, the cutoff scale is specified at the Planck energy, as mentioned in Subsec. 2.4. (Only as possible applications of mathematics to physics, Appendices provide a suppression of quadratic self-energies due to Higgs fields (around 10^{11} eV (10^2 GeV)) [24], which are extremely large for an inevitably large ultraviolet cutoff. At the energy lower than the ultraviolet cutoff energy, this cutoff is achieved by dividing spacetime continuum into arbitrarily shaped elements, based on solid-state physics [25, 26].)

The fields are Poincaré covariant under rotational and translational transformations in the spacetime continuum. The expansion coefficients of the Yang-Mills gauge field have a symmetry represented by such as a compact and simple group [2]. The four-momentum is continuous for energies lower than the cutoff energy. The Lagrangian in the action is defined in a traditional way for the cutoff scheme, and the fields are quantized using the path integral. The formalism that presents Theorem 2.12 constructs Poincaré/Lorentz covariant/invariant Yang-Mills fields without ultraviolet divergences, and is consistent with the corresponding properties of the axiomatic formalism [5] by Streater & Wightman (SW) or equivalently that by Osterwalder & Schrader (OS). It is emphasized that except for high-energy contributions, which give rise to the ultraviolet divergence problem, the formalism of physical field theory is already mathematically rigorous. The field theory defined in theoretical physics is therefore not repeated mathematically here.

Next, in Sec. 3, a solution is presented to the mass gap problem for pure Yang-Mills fields, stated by Theorem 3.28. The quantization scheme employed in this paper is the Feynman path integral. It is regarded that contributions to the path integral are a variational stationary classical field and quantum fields. The former classical field leads to a differential equation in analytic dynamics, and the latter quantum fields fluctuate around the classical field. The pure Yang-Mills field then comprises classical and quantum fields. An example of compact simple groups considered here is represented by such as $SU(N)$ of degree N , which does not involve any loss of generality. The classical localized field with $SU(2)$ symmetry is embedded in the classical $SU(N)$ field with all zero elements. The classical field considered in this study is the sum of localized and unlocalized fields. For a specific configuration of the classical fields, the classical Yang-Mills differential equation is then reduced to a linear equation. This linear equation differs from that for the $U(1)$ gauge field, because each component of the Yang-Mills field is multiplied by a generator (matrix) of the group. The charge and anticharge densities associated with the localized and unlocalized fields, respectively, cancel each other. The classical localized field gives rise to a linear potential between a charge and an anticharge. Quantum fluctuations around the vacuum of the classical zero-field provide a Coulomb potential between paired charges. By contrast, when the classical field contains the localized field, a linear potential appears, and quantum fluctuations add the Coulomb potential to the linear potential. The latter confinement phase, which yields the binding energy between the paired charges under the linear potential, is stable compared to the former Coulomb phase. Moreover, the non-interacting term in the Lagrangian of the action gains a positive local mass originating from the non-vanishing classical field. Thus, in Theorem 3.28, not only the above binding energy between the paired charges but also the local mass contributes to the mass gap of the pure Yang-Mills field. Hence, this paper presents a solution to the Yang-Mills existence and mass gap problem.

This paper is organized as follows. Subsections 2.1 and 3.1 outline Secs. 2 and 3, respectively, for easier understanding. Then, the next Sec. 2 presents a well-defined formulation of Yang-Mills fields without ultraviolet divergences, leading to Theorem 2.12. Subsequently, Sec. 3 gives a solution to the mass gap problem of the pure Yang-Mills field, resulting in Theorem 3.28. Although usual mathematical papers need no conclusion section, Sec. 4 in this paper summarizes conclusions. Appendices provides possible applications of mathematics to physics for cutoffs of physical phenomena containing self-energies.

2 A solution to the Yang-Mills existence problem: Formulation of quantum field theory without ultraviolet divergences

2.1 Aim and outline of Sec. 2

This Subsec. 2.1 describes an outline of this Sec. 2. Quantum field theory in physics is rigorous, except for the problem of ultraviolet divergences. This section then presents a Poincaré covariant field theory without ultraviolet divergences, leading to a solution to the Yang-Mills (YM) existence problem. The remainder of this subsection describes the background, aim, and outline of this section.

Quantum field theory, which is quantized equivalently by such as the path integral or canonical quantization, has too many high-energy configurations (values), resulting in the ultraviolet divergence problem. Therefore, the ultraviolet divergences must be removed. Using plane waves in Eq. (2.8) with coordinates x_μ and energy-

momentum (four-momentum) p_μ on the four-dimensional spacetime continuum, a field without the cutoff of high energies (including momentums) is expressed in a Fourier expansion form of Eq. (2.9), where T^a denote generators of the group such as $SU(N)$ with N being a general and meaningful natural number. When a quantum particle with unlimited energies emitted from a field is reabsorbed by the field in the self-interacting intermediate state, ultraviolet divergences occur in some physical quantities. For field theory, the Poincaré covariance with the rotational and translational covariance is required. It is noted that in this paper, the underlining for quantities with the cutoff is omitted, except in cases where it is explicitly required to denote the cutoff.

A cutoff for removing ultraviolet divergences is treated in Subsec. 2.4 in the main procedure of this paper, giving a mathematics framework. As mentioned in Sec. 1, Yang-Mills fields with asymptotic freedom results in the phenomena that interactions disappear and particles are free for the infinite value of the cutoff energy. The phenomena mean the triviality of the Yang-Mills theory for the infinite cutoff energy. To avoid this triviality, Subsec. 2.4 introduces a finite cutoff for Yang-Mills fields. In mathematics, it does not necessarily seem that the metric is not determined from the general relativity, as described below in Subsec. 2.2. Only if the metric from general relativity is treated, the metric yields the loss of possibilities of high-energy intermediate states into black holes. Only when this metric like Riemannian geometry is dealt with, the cutoff scale is restricted to the Planck scale. This cutoff prevents the loss of possibilities of high-energy intermediate states into black holes.

To provide an overview of the cutoff in quantum field theory calculations mentioned below, quantum theory used in this paper is briefly summarized here, with reference to the literatures [2–6, 9–15]. In principle, field theory is formulated using a differential equation reproduced from a stationary state with a vanishing variation of the action functional (Lagrangian) in analytical dynamics. One quantization theory is Feynman path integral, in which integrals are performed over all field configurations (field function values) associated with the weight of the action. Canonical quantization of the operator form is another theory that was developed from the Hamilton formalism in analytic dynamics. As reported in the literatures [13, 14], a state vector $|\Psi\rangle$, which represents a field system, changes to $|\Psi'\rangle$ by the multiplication of matrix U_I in interactions, as shown in Eq. (2.18) in Subsec. 2.5. The probability (transition) amplitude in Eq. (2.19) projected onto the vector $\langle\Psi''|$ is derived from $\langle\Psi''|\Psi'\rangle = \langle\Psi''|U_I|\Psi\rangle$. When a coupling constant in interactions is small like that of the Yang-Mills field with the well-known asymptotic freedom [2] mentioned later, an expansion of U_I is performed using the Feynman-Dyson diagram method. It is known that in path integral, derivatives of a generating functional yield terms in a power series of U_I with respect to a coupling constant [2, 3]. Using the Fourier expansion of fields as a function of four-momentum, field theory calculations are performed in momentum space. In the Feynman diagram, as shown later in Figure 1 in Subsec. 2.5, a straight or curved line connecting two intersections of lines physically corresponds to the vacuum expectation value for a time-ordered product of field operators (in the interaction representation). This expectation value, so-called such as the invariant function, propagator, Green function, correlation function, or Wightman function, is typically a tempered distribution. From the Wightman function in the Minkowski space, Schwinger function in the Euclidean space is derived through a transformation on a complex plane. In integrals in 4D momentum space according to the Feynman diagram, high-energy contributions to the diagrams, such as so-called loop and vertex, yield ultraviolet divergences. Proper diagrams, which cannot be essentially divided, are separated, including the case of a higher-order series. The ultraviolet divergences are removed by cutting off high-energy contributions. Notably, the characteristics of this paper employ the Lorentz-invariant cutoff

radius, which is rotationally (Lorentz) invariant in the Euclidean space. The divergent integrals described by the proper diagrams are put into a set of limited number of renormalization constants (including subtraction cases), and fields are renormalized. Due to this renormalization, the coupling constant depends on the cutoff energy. Yang-Mills fields then reveal the asymptotic freedom, which is the property that the coupling constant decreases as the cutoff energy increases.

The Poincaré covariant cutoff in this paper is used in the following procedure. The Feynman-Dyson diagram method developed in the original studies [12–14] are described in detail in many literatures [5, 15]. Figure 1 depicted later in Subsec. 2.5 shows a two-point graph for a self-interaction in the Feynman diagram, representing a two-point function below. In this graph, the process $x \rightarrow y$ is finally taken. Although this graph exactly corresponds to Eq. (2.29), the conceptual meaning is roughly symbolized in Eq. (2.20). The general term displayed by the Feynman diagram is given by Eq. (2.27), which can be rewritten as a simple product of the two-point functions, as indicated in Eq. (2.28). In a sample case, Eq. (2.28) becomes simple Eq. (2.29) followed by Eq. (2.30). In the diagram, a quantum particle with a momentum p (namely p_μ) annihilates at intersection y and a particle with momentum k is created. This particle propagates along a curved line and annihilates at intersection x . Here, a particle with the momentum p is created. In this case, the process $x \rightarrow y$ is finally taken to yield a quartic interaction, in which the field reabsorbs the emitted particle at the same intersection. The field particle without the cutoff propagates freely between the two intersections in the intermediate state during the interaction and takes any momentum k . Conversely, the momentum with the cutoff is restricted within the cutoff energy.

Here, the remainder of this Subsec. 2.1 sequentially overviews the theoretical procedures in the current Sec. 2, summarizing each subsection and referring to some equations.

Section 2.2 presents the metric of the 4-dimensional spacetime continuum, on which Yang-Mills fields are defined. In mathematics, if the metric is given like for Riemannian geometry, it is not necessarily excluded that the metric is determined from general relativity. This metric leads to black holes at the Planck scale, which results in unphysical missing of particles in intermediate states.

Subsection 2.3 describes a Poincaré covariant plane wave in Eq. (2.4), which is used for a Fourier expansion as a function of four-momentum, since the cutoff in this paper is also represented in 4D momentum space.

Subsection 2.4 presents the main procedure for a cutoff to avoid the triviality of the Yang-Mills theory for the infinite high energy. Only if the metric is given by general relativity, the cutoff energy is set to the Planck energy, preventing the particle probability loss of high-energy intermediate states into a black hole. A Yang-Mills theory on the 4D spacetime continuum is defined without ultraviolet divergences.

In Subsec. 2.5, the beginning up to around Eq. (2.30) is devoted to summarizing the Feynman-Dyson diagram method, which gives an expansion in powers of a coupling constant, for field theory calculations. A (vacuum) expectation value of a product of operators is given in the Minkowski space in Eq. (2.30). Using the residue theorem for complex functions, the expectation value is rewritten to the form in the Euclidean momentum space, as shown in Eq. (2.31). The field without the cutoff yields divergences of self-interaction quantities contained in the transition amplitude as shown in Eq. (2.34), while the field with the rotationally invariant cutoff gives a finite value without ultraviolet divergences, as indicated in Eq. (2.35).

Subsection 2.6 refers to the relationship between (vacuum) expectation values in the Minkowski spacetime

and tempered distributions, showing the energy cutoff.

The lemmas and the theoretical schemes in this Sec. 2 finally lead to Theorem 2.12, which is a solution to the Yang-Mills existence problem.

2.2 Spacetime continuum on which fields are defined

As mentioned in the previous Sec. 1, mathematical theories, which are independent of physics, are constructed in this paper, referring to physical formulations with their meanings. In this section 2, quantum field theory (QFT) without ultraviolet divergences is formulated by introducing a cutoff. Fields with state vectors are defined mainly on the four-dimensional (4D) Euclidean spacetime continuum \mathbb{R}_E^4 . This Subsec. 2.2 defines the metric of the 4D spacetime continuum. At the end of this subsection, it is added that in mathematics, only if the metric of the 4D spacetime continuum is determined from general relativity like for Riemannian geometry, particles in high-energy intermediate states at the Planck energy are lost into black holes.

Definition 2.1. For a set of real numbers \mathbb{R} , let $t \in \mathbb{R}$ and $x, y, z \in \mathbb{R}$ be time and space coordinates, respectively. Physically, $c = 1$ and $\hbar := h/\pi = 1$ in natural units, where c and h correspond to the speed of light and the Planck constant, respectively. Let $\mathbb{R}^4 = \{x^\mu\}$ with $\mu = 0, 1, 2, 3$ be 4D spacetime continuum, where x^μ are contravariant components of a four-vector, denoted by $x^0 := t = ct$, $x^1 := x$, $x^2 := y$, $x^3 := z$. It is noted that x used as a coordinate is also used as a four-vector denoted by $x = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}) = (x^0, \mathbf{r}) = (t, x, y, z)$. Using a repeated Roman index i with $i = 1, 2, 3$, the summation of spatial components is expressed by $(x^i)^2 := x_i x^i = (x^1)^2 + (x^2)^2 + (x^3)^2$. Hereafter, the covariant and contravariant components are not distinguished, because there is no need to distinguish between them in the Yang-Mills theory. Let \mathbb{R}_M^4 be the four-dimensional Minkowski spacetime continuum. For $x_\mu, y_\mu \in \mathbb{R}_M^4$ with the repeated Greek index μ , which is a non-negative integer that runs from 0 to 3, the scalar inner product is defined by $x \cdot y := x_\mu y_\mu = -x_0 y_0 + x_i y_i$ instead of $x \cdot y = x_0 y_0 - x_i y_i$. Via Wick rotation, this product is transformed to that in the Euclidean spacetime continuum denoted by $\mathbb{R}_E^4 = \mathbb{R}^4$. Then, for $x_\mu, y_\mu \in \mathbb{R}_E^4$, the scalar inner product and norm are defined by $x \cdot y := x_\mu y_\mu = x_0 y_0 + x_i y_i$ and $|x \cdot x| := ||x \cdot x|| := (x \cdot x)^{1/2}$, respectively.

In mathematics, the metric of the spacetime continuum can be included in an axiom as for Riemannian geometry. It does not necessarily seem that the metric of spacetime is not determined from general relativity. Only if the spacetime metric is provided by general relativity, the theory in this paper consider the following phenomena. It is known that the Compton wave length is the particle size, while the black hole size is equal to the size of the Schwarzschild sphere. When the Compton wave length of a particle is smaller than the Schwarzschild size, the particle becomes a black hole. In high-energy intermediate states, the particle misses the probability, when the particle becomes the black hole. This theoretical result is sufficient for mathematics, which does not need experimental facts for black holes, because mathematics is not physics and does not need the physical experimental evidence.

2.3 Poincaré invariance/covariance of plane waves

In this subsection, Poincaré invariant/covariant plane waves are considered. Using four-momentum p_μ , a plane waves is expressed as a function of an inner product $p_\mu x_\mu$ in the form defined by Eq. (2.4) below. The

plane wave is used for a Fourier expansion without and with the cutoff in the momentum space, as mentioned in Subsec. 2.1.

Lemma 2.2. For $x_{(1)}, x_{(2)}, x'_{(1)}, x'_{(2)} \in \mathbb{R}_E^4$, let $L_{\mu\nu}$ be a matrix that represents a Lorentz transformation. For vectors $x_{(2)} - x_{(1)}$ and $x'_{(2)} - x'_{(1)}$, the following Lorentz transformation is considered

$$x_{(2)} - x_{(1)} = x_{(2)\nu} - x_{(1)\nu} = L_{\nu\mu} (x'_{(2)\mu} - x'_{(1)\mu}). \quad (2.1)$$

Then for $x_{(1)}, x_{(2)}, x_{(3)}, x_{(4)} \in \mathbb{R}_E^4$ as well as vectors $x_{(2)} - x_{(1)}$ and $x_{(4)} - x_{(3)}$, the scalar inner product of these vectors is Lorentz covariant/invariant:

$$\begin{aligned} P_{1(43,21)} &:= (x_{(4)} - x_{(3)}) \cdot (x_{(2)} - x_{(1)}) \\ &= (x'_{(4)\mu} - x'_{(3)\mu})(x'_{(2)\mu} - x'_{(1)\mu}). \end{aligned} \quad (2.2)$$

Proof. The above lemma is simply shown as

$$\begin{aligned} P_{1(43,21)} &= L_{\nu\eta} (x'_{(4)\eta} - x'_{(3)\eta}) L_{\nu\mu} (x'_{(2)\mu} - x'_{(1)\mu}) \\ &= L_{\eta\nu}^{-1} (x'_{(4)\eta} - x'_{(3)\eta}) L_{\nu\mu} (x'_{(2)\mu} - x'_{(1)\mu}) \\ &= \delta_{\eta\mu} (x'_{(4)\eta} - x'_{(3)\eta}) (x'_{(2)\mu} - x'_{(1)\mu}), \end{aligned} \quad (2.3)$$

where $\delta_{\eta\mu}$ is Kronecker's delta. □

Lemma 2.3. For $x_{(1)}, x_{(2)} \in \mathbb{R}_E^4$ as well as a constant four-vector a_μ , let $x_{(1)\mu} = x'_{(1)\mu} + a_\mu$ and $x_{(2)\mu} = x'_{(2)\mu} + a_\mu$ be translations contained in the Poincaré transformation. Using the quantities in Lemma 2.2, and considering that $x_{(2)\mu} - x_{(1)\mu} = x'_{(2)\mu} - x'_{(1)\mu}$ and $P_{1(43,21)}$ in Eq. (2.2) are Poincaré covariant/invariant, it follows that a plane wave

$$\chi_p(x) := \exp(ip_\mu x_\mu), \quad (2.4)$$

is a scalar function for $x_\mu \in \mathbb{R}_E^4$ and a four-vector p_μ .

Proof. The above lemma is simply because in Lemma 2.3, the quantity x_μ can be regarded as a vector for $x_{(2)\mu} = x_\mu$ and $x_{(1)\mu} = (0, 0, 0, 0)$. Here, $x_{(4)\mu} - x_{(3)\mu}$ is regarded as p_μ in Eq. (2.4). □

2.4 The Poincaré invariant/covariant cutoff

Yang-Mills fields have a property of asymptotic freedom. This property results in that interactions disappear and particles become free for the infinite value of the cutoff energy. This fact leads to the triviality of the Yang-Mills theory for the infinite cutoff energy. A finite cutoff energy, which is not specified at the present stage, is therefore introduced for Yang-Mills fields. As aforementioned in Subsec. 2.2 for the metric of the 4-dimensional spacetime continuum on which Yang-Mills fields are defined, it does not necessarily seem that in mathematics, the metric of general relativity like Riemannian geometry and associated black holes are not treated. Only if the spacetime metric is determined from general relativity, the cutoff scale is specified at the Planck energy for avoiding the missing of particles into the black holes. It is noted that this paper does not state that black holes are related to the renormalization, since the renormalization is not a cutoff.

In the following, a special unitary group is considered as an example of a compact and simple group. This treatment does not involve any loss of generality of the obtained results for compact and simple groups.

Definition 2.4. For a meaningful natural number N , let us here consider a special unitary group $SU(2)$ or a more extended $SU(N)$ represented by matrices whose dimension is $N_D = N^2 - 1$, as an example of a compact and simple group. Let T^a be traceless Hermitian matrices defined as generators of the group, and generate the commutators

$$[T^a, T^b] := T^a T^b - T^b T^a = \sum_c i f^{abc} T^c, \quad (2.5)$$

where the indices a, b and c run from 1 to N_D , and f^{abc} are totally antisymmetric structure constants. It is noted that the summation of the quantities over these indices is explicitly denoted. Using parameters $\omega_a \in \mathbb{R}$, the $SU(N)$ group is represented by the following special unitary matrix

$$U(x) := \exp[ig \sum_a \omega_a(x) T^a], \quad (2.6)$$

where g denotes a coupling constant. For $x_v \in \mathbb{R}^4$, a Yang-Mills field $A_\mu(x)$ has the form of

$$A_\mu(x) = A_\mu(x_v) = A_\mu(t, \mathbf{x}) := \sum_a A_\mu^a(x) T^a. \quad (2.7)$$

It is also noted that each component $A_\mu^a(x)$ of the Yang-Mills field $A_\mu(x)$ is multiplied by the generator T^a of the group.

Definition 2.5. Let $\chi_p(x)$ be plane waves denoted by

$$\chi_p(x) = \exp(ip_\mu x_\mu), \quad (2.8)$$

where the momentum p_μ takes any value in the 4D momentum space. Then, using Definition 2.4, the quantum Yang-Mills field is expressed in the Fourier expansion form

$$A_\mu(x) = \sum_a \sum_p A_{\mu p}^a T^a \chi_p(x). \quad (2.9)$$

Whereas, using the notation

$$\underline{\chi}_{\underline{k}}(x) = \exp(i \underline{k}_\mu x_\mu), \quad \text{with } |\underline{k}| \leq \Lambda_{CO}, \quad (2.10)$$

as well as Definition 2.4, the quantum Yang-Mills field, whose momentums are restricted to $|\underline{k}| \leq \Lambda_{CO}$ due to the cutoff at Λ_{CO} in the Poincaré covariant form, is represented as follows:

$$\underline{A}_\mu(x) := \sum_a \sum_{\underline{k}} \underline{A}_{\mu \underline{k}}^a T^a \underline{\chi}_{\underline{k}}(x) \quad \text{with } |\underline{k}| \leq \Lambda_{CO}. \quad (2.11)$$

It is noted that since the expansion functions $\chi_k(x)$ are scalar plane waves in Subsec. 2.3, the individual coefficients $A_{\mu k}^a T^a$ transform as a vector and a tensor under a transformation of the Lorentz group and $SU(N)$ (and similar groups), respectively.

Definition 2.6. According mainly to the notations in the literature [2] for the Yang-Mills field, the Lagrangian density of the pure Yang-Mills field is defined by

$$\begin{aligned}\mathcal{L}_{\text{YM}} &:= -\frac{2}{4}\text{Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)] \\ &= -\frac{1}{4}\sum_a F_{\mu\nu}^a(x)F_{\mu\nu}^a(x),\end{aligned}\tag{2.12}$$

where Tr is trace of the matrix, and

$$F_{\mu\nu}^a := \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) - g \sum_{b,c} f^{abc} A_\mu^b(x) A_\nu^c(x).\tag{2.13}$$

Although this paper focuses on pure Yang-Mills fields, it briefly summarizes that the Yang-Mills field is a gauge field. A fermion field $\psi(x)$ comprises components $\psi^a(x)$, and the Lagrangian density of fermions with a mass of nearly zero is denoted as

$$\mathcal{L}_F := -\psi^* \gamma_4 \gamma_\mu \partial_\mu \psi(x),\tag{2.14}$$

where γ_μ are Dirac matrices. For the gauge transformation of the fermion field

$$\psi(x) \rightarrow U(x)\psi(x),\tag{2.15}$$

the covariant derivative is defined by

$$D_{\mu\nu} := \partial_\mu + igA_\mu(x).\tag{2.16}$$

Then, the gauge transformation of the Yang-Mills field is expressed as

$$A_\mu(x) \rightarrow U(x)A_\mu(x)U^{-1}(x) - \frac{i}{g}U(x)\partial_\mu U^{-1}(x).\tag{2.17}$$

2.5 Vacuum expectation value for a product of operators with the cutoff

As aforementioned in Subsec. 2.1, this subsection first describes the Feynman diagram method. Subsequently, corresponding physically to a vacuum expectation value for a product of operators, an expectation value for a quantity is presented, in the case without and with the cutoff. The established Feynman diagram method has been described in detail in many literatures [2, 4–6, 12–15], and this paper briefly summaries this method. The integral for the expectation value in the Minkowski momentum space is rewritten in the Euclidean momentum space using the residue theorem on a complex plane.

A state vector $|\Psi\rangle$, which describes a quantum Yang-Mills system, transit to another state $|\Psi'\rangle$ by multiplication by a matrix U_I , as follows:

$$|\Psi'\rangle = U_I |\Psi\rangle.\tag{2.18}$$

The probability (transition) amplitude projected to the vector $\langle\Psi''|$ is given by

$$\langle\Psi''|\Psi'\rangle = \langle\Psi''|U_I|\Psi\rangle.\tag{2.19}$$

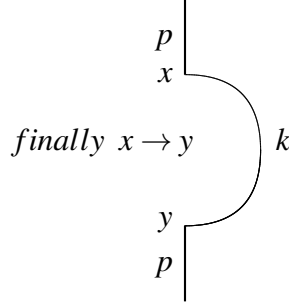


Figure 1: A Feynman diagram for a self-interaction of a field.

The above U_1 is expanded by a power series of a coupling constant g , and the general term of the series is depicted using the Feynman diagram, as shown in Fig. 1. This figure is an example of a graph, which describes a self-interaction of the field, with final setting of $x \rightarrow y$. In this interaction, a quantum particle named gluon with an initial four-momentum p annihilates at the intersection y of two (straight or curved) lines, followed by the creation of a particle with momentum k . The particle with k is annihilated at the intersection x , and a particle with the momentum p is created. The straight or curved line indicates the field-particle propagation without momentum change. In canonical quantization, at the intersection of straight/curved lines, particles with each momentum are created/annihilated due to creation/annihilation operators and the associated coupling constant. The intersection is a location where the momentum changes. It is known that path integral quantization without creation and annihilation operators is equivalent to canonical quantization and provides an identical expectation value. Finally, by putting $x \rightarrow y$, this process becomes the case, in which emitted particle is reabsorbed at the same intersection. In quantum mechanics, the absolute momentum $|k|$ in this intermediated state takes any value from zero to infinity. The process between the intersections x and y is the interaction in quantum theory. The freedom of the momentum k for this self-interaction in the intermediate state is excessively large; and therefore, the integral over the momentum k results in an ultraviolet divergence. The solution in this paper prevents the ultraviolet divergence by the Poincaré invariant/covariant cutoff, which removes contributions from high energies beyond the cutoff energy, and enables us to treat fields without ultraviolet divergence.

Before describing the expression depicted by the Feynman diagram in Fig. 1, the interaction of one component of the Yang-Mills fields is roughly written as

$$\begin{aligned} &\approx \delta_{\mu\mu'} g^2 \int d^3k \left\{ \frac{1}{(2\pi)^3 2k_0} a_{\mu'}^{*a} \exp(-ipx) \theta(x_0 - y_0) \exp[ik(x - y)] \right. \\ &\quad \left. \times <0|a_{\mu}^a(\mathbf{k}) a_{\mu'}^{*a}(\mathbf{k})|0> a_{\mu'}^a \exp(ipx) \right\}, \end{aligned} \quad (2.20)$$

where $a_{\mu}^{*a}(\mathbf{k})$ and $a_{\mu}^a(\mathbf{k})$ are creation and annihilation operators of a quantum particle of the Yang-Mills field

for the component a . Here,

$$\theta(t) := \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}, \quad (2.21)$$

with $\theta(0) := 1/2$. For the rough Eq. (2.20), the strict expression is derived below, noting that the quantities with the repeated group index a are not summed. The fields without the cutoff have any four-momentums, including those beyond the Planck energy. The Yang-Mills field $A_\mu(x) = T^a A^a(x)$ without the cutoff is expressed in the form of a Fourier expansion

$$A_\mu^a(x) = \int d^3k \frac{1}{\sqrt{(2\pi)^3 2k_0}} [a_\mu^a(\mathbf{k}) \exp(ik_\mu x_\mu) + a_\mu^{*a}(\mathbf{k}) \exp(-ik_\mu x_\mu)], \quad (2.22)$$

where k_μ is the energy-momentum, and \mathbf{k} is the three-dimensional momentum. The field without cutoff has some difficulties. In the quantum field scheme, an interacting field exchanges a quantum particle with any momentum in the intermediate state. The field with the four-momentum k_μ within the cutoff energy is denoted by $\underline{A}_\mu(x) = T^a \underline{A}^a(x)$. For $\underline{A}_\mu(x)$, the field expression corresponding to Eq. (2.22) becomes

$$\underline{A}_\mu^a(x) = \int d^3\underline{k} \frac{1}{\sqrt{(2\pi)^3 2\underline{k}_0}} [a_\mu^a(\underline{\mathbf{k}}) \exp(i\underline{k}_\mu x_\mu) + a_\mu^{*a}(\underline{\mathbf{k}}) \exp(-i\underline{k}_\mu x_\mu)]. \quad (2.23)$$

Hereafter, only when necessary, the underlined symbols such as \underline{k}_μ and $\underline{A}_\mu(x)$ with the cutoff are explicitly indicated, and most field quantities with the energy cutoff are not distinguished from those without the cutoff. Using the symbol $[A, B] = AB - BA$, the creation and annihilation operators satisfy

$$[a_\mu^a(\mathbf{k}), a_\nu^{*b}(\mathbf{k}')] = \delta_{ab} \delta_{\mu\nu} \delta_{\mathbf{k}\mathbf{k}'}, \quad [a_\mu^a(\mathbf{k}), a_\nu^b(\mathbf{k}')] = 0, \quad [a_\mu^{*a}(\mathbf{k}), a_\nu^{*b}(\mathbf{k}')] = 0, \\ a_\mu^a(\mathbf{k})|0\rangle = 0. \quad (2.24)$$

Additionally, T product, which is the time-ordered product of operators, is defined by

$$TA_\mu^a(x)A_\nu^b(y) = \theta(x_0 - y_0)A_\mu^a(x)A_\nu^b(y) + \theta(y_0 - x_0)A_\nu^b(y)A_\mu^a(x), \quad (2.25)$$

using Eq. (2.21).

For a positive natural number n , the coordinates x, y are generalized to $x_{(1)}, \dots, x_{(n)}$, and the abbreviated quantities

$$b(x) := A_\mu^a(x), \quad b(y) := A_\nu^b(y), \quad (2.26)$$

are generalized to $b(x_{(1)}), \dots, b(x_{(n)})$. For a positive natural number j , it is known that U_I in Eq. (2.18) is expanded into a power series with respect to the coupling constant g , and each term is depicted by a Feynman graph. This term is written in the following form, which is further multiplied by a power of g and structure constants of the group,

$$\sum_{\text{comb}} [Nb(x'_{(1)}) \cdots b(x'_{(j)})] < 0 | Tb(x'_{(j+1)}) \cdots b(x'_{(n)}) | 0 >, \quad (2.27)$$

where \sum_{comb} denotes the sum over all combinations of $x_{(1)}, \dots, x_{(n)}$. The quantity $Nb(x'_{(1)}) \cdots b(x'_{(j)})$ is defined as N product in which creation operators $a_\mu^{*a}(\mathbf{k})$ and annihilation operators $a_\mu^a(\mathbf{k})$ are placed on the

left and right, respectively. For odd n , the above expectation value $\langle 0|Tb(x'_{(j+1)}) \cdots b(x'_{(n)})|0 \rangle$, which physically corresponds to the vacuum expectation value, vanishes, while for even n , the value is rewritten as

$$\begin{aligned} & \langle 0|Tb(x'_{(j+1)})b(x'_{(j+2)}) \cdots b(x'_{(n)})|0 \rangle \\ &= \sum_{k_1 < k_2, k_3 < k_4, \dots} \langle 0|Tb(x'_{(k_1)})b(x'_{(k_2)})|0 \rangle \cdots \langle 0|Tb(x'_{(k_{n-1})})b(x'_{(k_n)})|0 \rangle, \end{aligned} \quad (2.28)$$

where k_1, k_2, \dots, k_n are permutations of $j+1, j+2, \dots, n$. Using Eqs. (2.22)–(2.25) denoted by the original notations, such as $A_\mu^a(x)$ for $b(x'_{(k_1)})$ in Eq.(2.26), the above (vacuum) expectation value multiplied by g^2 is given by

$$\begin{aligned} & g^2 \langle 0|TA_\mu^a(x)A_\nu^b(y)|0 \rangle \\ &= g^2 \delta_{ab} \int d^3k d^3q \left\{ \frac{1}{\sqrt{(2\pi)^3 2k_0}} \frac{1}{\sqrt{(2\pi)^3 2q_0}} \right. \\ & \quad \times [\theta(x_0 - y_0) \exp(ikx - iqy) \langle a_\mu^a(\mathbf{k})a_\nu^{*b}(\mathbf{q})|0 \rangle \\ & \quad \left. + \theta(y_0 - x_0) \exp(-ikx + iqy) \langle a_\nu^b(\mathbf{q})a_\mu^{*a}(\mathbf{k})|0 \rangle \right\} \\ &= g^2 \delta_{ab} \delta_{\mu\nu} \int d^3k \left\{ \frac{1}{(2\pi)^3 2k_0} \right. \\ & \quad \left. \times [\theta(x_0 - y_0) \exp(ik(x - y)) + \theta(y_0 - x_0) \exp(-ik(x - y))] \right\}. \end{aligned} \quad (2.29)$$

Now, Eq. (2.29) is expressed in the following integral with the help of the residue theorem of complex analysis:

$$\begin{aligned} & g^2 \langle 0|TA_\mu^a(x)A_\nu^b(x)|0 \rangle \\ &= \delta_{ab} \delta_{\mu\nu} g^2 \int d^4k \frac{1}{i(2\pi)^4} \frac{1}{k^2 + (m_\varepsilon)^2 - i\varepsilon} \exp[ik(x - y)], \end{aligned} \quad (2.30)$$

where ε is an infinitesimal value, and the limits of $m_\varepsilon \rightarrow 0$, $\varepsilon \rightarrow 0$ are taken after calculations. Let us extend the real number k_0 to the complex number shown in Fig. 2. In the integral $\int_{-\infty}^0 dk_0(\cdots)$ on the real axis, the pole indicated by \times that lies at $-\sqrt{\mathbf{k}^2 + (m_\varepsilon)^2} - i\varepsilon = -\sqrt{\mathbf{k}^2 + (m_\varepsilon)^2} + i\varepsilon'$ infinitesimally above the real axis. Here, ε' is also an infinitesimal quantity, which is taken as $\varepsilon' \rightarrow$ after calculations. Similarly, for the integral $\int_0^{+\infty} dk_0(\cdots)$ on the real axis, the pole denoted as \times that lies at $+\sqrt{\mathbf{k}^2 + (m_\varepsilon)^2} - i\varepsilon = +\sqrt{\mathbf{k}^2 + (m_\varepsilon)^2} - i\varepsilon'$ infinitesimally below the real axis. For the field without the cutoff for $x_0 - y_0 > 0$, we add the vanishing integral on the lower half circle infinitely distant from the origin as shown in Fig. 2. For $x_0 - y_0 < 0$, we also add the integral on the upper half circle. Based on the residue theorem, Eq. (2.30) reproduces Eq. (2.29). For the field with the cutoff, the outcome is the same as that for the case without the cutoff, because the cutoff leads to vanishing of the function of the momentum in the region that is sufficiently distant from the origin.

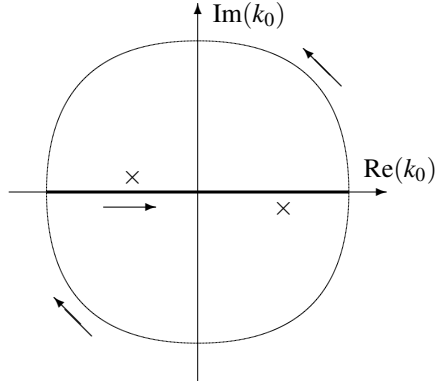


Figure 2: Contour of the integral for a (vacuum) expectation value using the residue theorem.

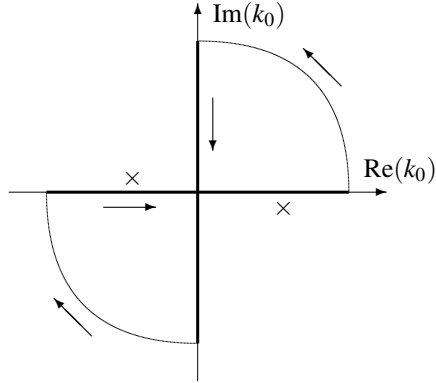


Figure 3: Contour for the energy integral on the real axis in Minkowski space and rotation to that on the imaginary axis.

Next, the integral above is transformed to the Euclidean integral using the residue theorem. As mentioned below, the integral path is rotated by $\pi/2$, as shown in Fig. 3. For k_0 on the complex plane, the integral $\int_{-\infty}^{+\infty} dk_0(\dots)$ on the real axis is extended by adding the vanishing integrals $\oint_0^{\pi/2}$ (which is rather symbolic) and $\oint_{-\pi/2}^{-\pi}$ on an infinitely large radius circle. Furthermore, we also add the integral on the imaginary axis $\int_{+i\infty}^{-i\infty} dk_0$. Then, based on the residue theorem, the integral on the real k_0 axis becomes that over the imaginary ik_0 axis with k_0 being real (k_0 is the limited meaning used here only). The quantity $-(k_0)^2$ in the integrand also becomes $(k_0)^2$ that is equivalent to the Euclidean metric. This implies that using the contour rotation,

the integral with the Minkowski metric is transformed to that with the Euclidean metric. From Eq. (2.30), the Euclidean integral in the limit of $m_\varepsilon \rightarrow 0$, $\varepsilon \rightarrow 0$, $\varepsilon' \rightarrow 0$ yields

$$g^2 < 0 |TA_\mu^a(x)A_\nu^b(x)|0> = g^2 \delta_{ab} \delta_{\mu\nu} \int d^4k \frac{1}{(2\pi)^4} \frac{1}{k^2} \exp[ik(x-y)]. \quad (2.31)$$

The above (vacuum) expectation value has the form of a Fourier series as a function of momentum in the Euclidean space. It is known that the transition amplitude in Eq. (2.19) contains self-interaction quantities expressed by the integral over momentums for the individual (vacuum) expectation value. An example of the self-interaction is depicted by a Feynman graph in Fig. 1.

Lemma 2.7. *A self-interaction quantity contained in the transition amplitude in Eq. (2.19) diverges in the case without the cutoff, while the self-interaction quantity has a finite value in the case with the cutoff.*

Proof. A self-interaction quantity depicted by such as a graph in Fig. 1 has the form

$$\frac{1}{k^2}. \quad (2.32)$$

In the Euclidean-space integral, the following relation is used

$$d^4k = dk \, 2\pi^2 k^3 = d(k^2) \, \pi^2(k^2). \quad (2.33)$$

For the field without the cutoff, the integral in Eq. (2.31) with respect to k^2 over all momentum values amounts to

$$\int d(k^2) k^2 \frac{1}{k^2} = \infty, \quad (2.34)$$

which is referred to as the ultraviolet divergence.

Whereas, for the field with the cutoff at the rotation-invariant Λ_{CO} , the momentum in the region within the cutoff value is here explicitly denoted as \underline{k} in Eq. (2.23) instead of k . In this case, the above integral is derived to give

$$\int_0^{(\Lambda_{\text{CO}})^2} d(\underline{k}^2) \underline{k}^2 \frac{1}{\underline{k}^2} = (\Lambda_{\text{CO}})^2, \quad (2.35)$$

which is proportional to the finite $(\Lambda_{\text{CO}})^2$ and the ultraviolet divergence is removed.

□

The above integral for the multi-component gauge field is multiplied by the antisymmetric structure constants. If the absolute value of one term has the same value as that of another term with the opposite sign, these terms are cancelled.

2.6 Relationship between vacuum expectation values of operator products and tempered distributions

This subsection describes the relationship between (vacuum) expectation values in the Minkowski spacetime and tempered distributions. The procedure in the present Sec. 2 finally derives Theorem 2.12, which presents a solution to the Yang-Mills existence problem. A mathematical expectation value for a product of operators is here modified using the cutoff at an energy E_{co} . For a proof of the lemma below, one can refer to the literatures [4, 5]. Because vector fields are expanded in terms of scalar plane fields, the essence of vector fields is obtained from the scalar fields.

Definition 2.8. For $x \in \mathbb{R}_M^4$, let $\phi(x)$ be a massless scalar field, which satisfies $(-\partial_t \partial_t + \partial_i \partial_i)\phi(t, \mathbf{r}) = 0$ in the no interaction case. In the interaction picture, the expectation value for a product of two-point operators $\phi(x)\phi(y)$ is denoted by $\langle 0 | T\phi(x)\phi(y) | 0 \rangle$, where $T\phi(x)\phi(y)$ is a time-ordered product (T product) of operators defined by

$$T\phi(x)\phi(y) := \begin{cases} \phi(x)\phi(y) & \text{for } x_0 > y_0 \\ \phi(y)\phi(x) & \text{for } x_0 < y_0 \end{cases}. \quad (2.36)$$

Lemma 2.9. According to the literature [5], the expectation value $\langle 0 | T\phi(x)\phi(y) | 0 \rangle$ in Definition 2.8 is given by

$$\langle 0 | T\phi(x)\phi(y) | 0 \rangle = \begin{cases} i \frac{-i}{2(2\pi)^3} \int_{k_0 > 0} d^3 k \frac{\exp[ik_\mu(x_\mu - y_\mu)]}{k_0} & \text{for } x_0 > y_0 \\ -i \frac{i}{2(2\pi)^3} \int_{k_0 > 0} d^3 k \frac{\exp[-ik_\mu(x_\mu - y_\mu)]}{k_0} & \text{for } x_0 < y_0 \end{cases}. \quad (2.37)$$

The detailed proof of the above relationship is described in the literature [5] by Schweber. A slightly different representation and derivation procedure for general fields were presented in Eq. (2.29).

Lemma 2.10. One of the above integrals yields

$$\frac{i}{2(2\pi)^3} \int_{k_0 > 0} d^3 k \frac{\exp(ik_\mu x_\mu)}{k_0} = \frac{4\pi}{r} \delta(r - t). \quad (2.38)$$

Proof. As written in the literatures by Tomonaga and coworkers (N. Fukuda, H. Fukuda, and K. Sawada) in Japanese as well as by Heitler [4], the above integral is reduced to

$$\begin{aligned} & \frac{i}{2(2\pi)^3} \int_{k_0 > 0} d^3 k \frac{\exp(ik_\mu x_\mu)}{k_0} \\ &= \frac{2\pi i}{(2\pi)^3} \int_0^\infty dk \frac{k}{2} \int_0^\pi d\theta \sin \theta \exp(ikr \cos \theta - ikt) \\ &= \frac{4\pi}{(2\pi)^3 r} \int_0^\infty dk \sin kr \sin kt = \frac{1}{8\pi^2 r} \int_{-\infty}^\infty dk \exp[ik(r - t)] \\ &= \frac{1}{4\pi r} \delta(r - t). \end{aligned} \quad (2.39)$$

□

The above physical δ -function corresponds to a mathematical tempered distribution, and indicates a signal propagation. However, the above integral causes ultraviolet divergences. The theoretical framework of this paper removes ultraviolet divergences using the cutoff. For the above Eq. (2.39), the below Eq. (2.46) is to be given as a simple example of the cutoff.

Corresponding physically to singular invariant functions in field theory, mathematical tempered distributions, which are included in Schwartz distributions, are usually utilized with the benefit that Fourier transformations are available. A simple example of the physical $\delta(x)$ -function, which was introduced by Dirac, is redefined mathematically as one of tempered distributions. Let us consider a test function $f_T: \mathbb{R} \rightarrow \mathbb{R}$. This is generally written by $f_T: U \rightarrow \mathbb{R}$ with U being an open set in \mathbb{R} ; and \mathbb{R} is generalized to \mathbb{R}^4 . For $\alpha, \alpha^\mu, \beta, \in \mathbb{N}$ with \mathbb{N} being a set of non-negative natural numbers in this case, higher-order partial derivative is denoted by $D^\alpha f_T(x) := \partial_x^\alpha f_T(x)$. The differentiation is generally expressed as $D^\alpha := \partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ with $\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$, and the derivative satisfies

$$\sup_{x \in \mathbb{R}} |x^\beta D^\alpha f_T(x)| < \infty, \quad (2.40)$$

where the symbol \sup denotes the supremum. Then a set of the test functions f_T which have the above properties is denoted by D . The Dirac delta function $\delta(x) \in D'$, with D' being a set of the tempered distributions, is introduced using the following scalar inner product with $f_T \in D$

$$\int dx \delta(x) f_T(x) := f_T(0). \quad (2.41)$$

More generally, $s(x) \in D'$ is introduced using the following scalar inner product with the test function $f_T \in D$

$$\langle s, f_T \rangle := \int dx s(x) f_T(x). \quad (2.42)$$

As is known, derivatives as well as the Fourier and inverse transforms are mathematically defined similarly.

The Fourier transform of the δ -function yields

$$\frac{1}{\sqrt{2\pi}} \int dx \exp(-ipx) \delta(x) = \frac{1}{\sqrt{2\pi}}. \quad (2.43)$$

The inverse Fourier transform of the above function is then

$$\frac{1}{\sqrt{2\pi}} \int dp \exp(ipx) \frac{1}{\sqrt{2\pi}} = \delta(x). \quad (2.44)$$

Lemma 2.11. *Let E_{co} be the cutoff energy. For simplicity, with the cutoff of solely energies for the integral $\int_{-\infty}^{\infty} dk \exp[ik(r-t)]$ in Eq. (2.39) is regularized to give*

$$\int_{-E_{co}}^{E_{co}} dk \exp[ik(r-t)] = \frac{2 \sin[E_{co}(r-t)]}{r-t}. \quad (2.45)$$

Proof. The integral above results in

$$\begin{aligned}
& \int_{-E_{\text{co}}}^{E_{\text{co}}} dk \exp[ik(r-t)] \\
&= \frac{1}{i(r-t)} \{ \exp[iE_{\text{co}}(r-t)] - \exp[-iE_{\text{co}}(r-t)] \} \\
&= \frac{2 \sin[E_{\text{co}}(r-t)]}{r-t}.
\end{aligned} \tag{2.46}$$

□

It is noted that the numerator of the above integral is written in the form of $2E_{\text{co}}(r-t) + \text{higher-order terms}$, and the integral has a finite value at $r-t=0$. When the scalar field has a mass μ_{m} , the following integral in Eq. (2.39)

$$\int_0^\infty dk \sin kr \sin kt \tag{2.47}$$

is replaced by

$$\frac{\partial}{\partial r} \int_0^\infty dk \frac{1}{\sqrt{k^2 + \mu_{\text{m}}^2}} \cos kr \sin k_0 t, \tag{2.48}$$

according to the literature [5].

The procedure without ultraviolet divergences in this Sec. 2 leads to the following theorem.

Theorem 2.12. *By introducing a cutoff with Poincaré covariance/invariance, the formalism in this paper defines pure Yang-Mills fields without ultraviolet divergences. Since vector fields are expanded in terms of scalar plane waves, the expansion coefficients of the fields are transformed as vectors under a Lorentz transformation. At the same time, each component of the Yang-Mills and its generalized fields is multiplied by the generator of the $SU(N)$ group, which is an example of compact and simple groups. Each of these fields constructs a covariant derivative as a gauge field with fermion fields, thereby forming an invariant/covariant system under the gauge transformation via a unitary matrix in terms of the group generators. This is an answer to the Yang-Mills existence problem of Millennium problem.*

It is noted that by removing the high-energy contributions, the present formalism is well-defined and consistent with the corresponding properties of the axiomatic formalism [5] by Streater & Wightman (SW) or equivalently Osterwalder & Schrader (OS), because these axioms are constructed based on physical fields such as Yang-Mills fields.

At the end of this section, the author adds the following remark.

Remark 2.13. The author provided descriptions in this paper based on or/and considering theories in the literatures [2–15, 21–29] including author's/authors' papers [16–20], as well as books written in Japanese by such as Tomonaga (with S. Fukuda H. Fukuda, and Sawada), Yukawa, Nakanishi, Kugo, Nishijima, Ezawa (with Watanabe, Suzuki, and Tasaki), Iwasaki, with a literature by Hara (T. Hara), and so on. If necessary,

expressions in this paper are described using the corresponding expressions in these literatures and my own contemporary books written on such as detailed lattice gauge theory [30], canonical quantization with Yang-Mills theory [31], and gauge theories in the standard model including Yang-Mills theory [32]. The present and associated papers by the author are referred to by a paper [33] (regarded as a private communication or peer-reviewed in some sense at present), and it seems that the contents in the present paper are readable for field theorists. A feature of the theory in this paper is that fields are defined on the spacetime continuum with limited degrees of freedom. The infrared divergence in the paper [34] is followed by such the literature [35] by CERN. Historically, infrared divergences are regarded as the phenomena that do not destroy the field-theoretic scheme, as seen in the article [36] (private communication). Additionally, the Yang-Mills fields are not regarded to be massless. The mass gap is shown in Theorem 3.28 given in Sec. 3.

Since experimental strong interactions are described by Yang-Mills fields, there exists at least one corresponding mathematical non-trivial Yang-Mills theory. Hence, the construction of a Yang-Mills theory can be presented by a suitably described paper.

3 A solution to the mass gap problem for pure Yang-Mills fields

3.1 Aim and outline of Sec. 3

This section presents a solution to the mass gap problem. This solution provides a property that although the Yang-Mills fields are gauge fields, these fields are not long ranged because of a non-vanishing mass. Furthermore, it is shown in a physical meaning that a pair of a quantum particle (gluon) and an antiparticle of the Yang-Mills field is strongly bound, making the pair separation difficult. This binding energy of the pair is equal to a non-vanishing mass of the field. In the case of quarks mediated by the Yang-Mills field, it is experimentally shown by the Regge trajectory that the long-range potential associated with the force between a quark and an antiquark is linear. Whereas, the short-range interaction in the same quark case reveals that the particles move rather freely under the Coulomb potential. Based on these experimental results, this section shows that the potential between a gluon particle and an antiparticle of a pure Yang-Mills field is a combination of linear and Coulomb potentials, similarly to the quark case. The Feynman path integral, a field quantization formalism, calculates an expectation value of a quantity, by summing probability values over all field configurations associated with the individual weight of the action functional. The action is expressed by an integral of the Lagrangian density on the spacetime continuum. This quantization is regarded as fluctuations from a stationary state derived from a vanishing variation in analytic dynamics. The stationary state is described by a differential field equation, which provides a solution of a classical field. This approach to a non-perturbative stationary state is similar to the intermediate-coupling theory [27] formulated by Tomonaga and coworkers. The solution resembles to a Laurant series containing both a term $1/z$ and the remainder expressed by a power series in complex analysis, where z is a complex number. Therefore, in this paper, a solution to the pure Yang-Mills field equation comprises a classical field and quantum fluctuations. The classical field is a base of the quantum fluctuations, and corresponds physically to a vacuum. Components of the classical field transform under transformations of such as the $SU(N)$ group including the $SU(2)$ subgroup, whose finite fields are embedded in an extended group with classical zero fields. An example of such an extended group is $SU(N)$ of the compact and simple group. By contrast, quantum fluctuations around the classical field are formulated by using the path integral of the fields with a

symmetry represented by the extended group. No restriction is imposed on the quantum path integral, unlike the classical field case.

The solution to the differential equation of the Yang-Mills equation is restricted to the form given by Eq. (3.1). For this type of solution, the non-linear field strength tensor in Eq. (3.3) becomes linear. This linear solution is not a U(1) field but an SU(2) field, because the component of the field is multiplied by the generator T^a of the group. It is known that the potential between a particle and an antiparticle is derived from the Wilson loop in Eq. (3.45). In order to derive a linear potential in Eq. (3.66), a classical localized function $A_{(CL)}(x)$ is expressed in the form given by Eq. (3.1). This classical localized field does not satisfy the Yang-Mills equation. By adding an unlocalized field $A_{(CU)}(x)$ in Eq. (3.28) to the localized field $A_{(CL)}(x)$ (with the charge density distribution such as that in Eq. (3.32)), the total field $A_{(CL)}(x) + A_{(CU)}(x)$ satisfies the Yang-Mills field equation given by Eq. (3.7). This is because the charges of these fields cancel each other. As described in Lemma 3.11, the added unlocalized field $A_{(CU)}(x)$ differs from the localized field, and does not influence the Wilson loop, as shown in Eq. (3.49). In the next procedure, the string tension (energy per unit length) for the linear potential is related to the size of the particle-antiparticle gluon pair, as indicated in Eq. (3.76). The decrease/increase in the energy of the pair is equal to the increase/decrease in the energy of the Yang-Mills field, as given in Eq. (3.74). The energy of the Yang-Mills field is derived from the energy-momentum tensor of the field in Eq. (3.68). In contrast to the atomic Bohr radius, which is obtained from fundamental physical units, the size of the pair of the particles is not determined, because this size is a given fundamental physical quantity. The binding energy at finite temperatures is derived from the Polyakov line in Eq. (3.78), and yield Eq. (3.81). This paper also shows that the Lagrangian density in the action functional has a local mass due to the included stationary field, as stated in Lemma 3.24. Finally, since quantum fluctuations for a sufficiently large cutoff energy lead to the asymptotic freedom with a small coupling constant, the potential derived from the Wilson loop reveals a Coulomb potential, as shown in Subsubsec 3.3.2.

Consequently, a solution to the mass gap problem is presented in Theorem 3.28.

The contents of this Sec. 3 are as follows.

- [3.2] Confinement for Yang-Mills particles of gluons
 - [3.2.1] A classical solution of pure Yang-Mills fields
 - [3.2.2] Classical Wilson loop
 - [3.2.3] Scale-invariant energy of Yang-Mills fields, string tension of the linear potential, and Polyakov's binding energy
- [3.3] Quantum field in path integral around the classical field as a vacuum
 - [3.3.1] Quantum action of pure Yang-Mills fields expanded in terms of scalar plane wave functions
 - [3.3.2] Quantum Wilson loop

3.2 Confinement for Yang-Mills particles of gluons

3.2.1 A classical solution of pure Yang-Mills fields

In the following, it is shown that the field expressed in Eq. (3.1) satisfies Eq. (3.7) without losing the properties of the $SU(N)$ group, because the field is multiplied by the generator of the group.

Definition 3.1. For $x \in \mathbb{R}_E^4$, let $A_{(C)\mu}(x)$ be a solution to the field equation of the classical pure Yang-Mills field. Here, using Definition 2.4, the present solution considered is that of $SU(2)$ group embedded in $SU(N)$ group in the form

$$A_{(C)\mu}(x) := \begin{cases} A_{(C)\mu}^a(x)T^a = \lambda^a \tilde{A}_{(C)\mu}(x)T^a & \text{for } a = 1, 2, 3 \\ 0 & \text{for } a > 3 \end{cases}, \quad (3.1)$$

where λ^a are real-number constants.

Lemma 3.2. For the classical Yang-Mills field in Eq. (3.1) with the structure constant f_{abc} in Definition 2.4, let's consider the following field strength

$$F_{(C)\mu\nu}^a := \partial_\mu A_{(C)\nu}^a - \partial_\nu A_{(C)\mu}^a - g_c \sum_{b,c} f^{abc} A_{(C)\mu}^b A_{(C)\nu}^c, \quad (3.2)$$

where g_c is a classical coupling constant. Then it follows that

$$F_{(C)\mu\nu}^a = \partial_\mu A_{(C)\nu}^a - \partial_\nu A_{(C)\mu}^a. \quad (3.3)$$

Proof. Since algebraic structure constants of the $SU(N)$ group are totally antisymmetric denoted as $f_{abc} = -f_{acb}$, the following self-interaction term in Eq. (3.2) vanishes, to give

$$g_c \sum_{b,c} f^{abc} A_{(C)\mu}^b A_{(C)\nu}^c = g_c \tilde{A}_{(C)\mu} \tilde{A}_{(C)\nu} \sum_{b,c} \lambda^b \lambda^c f^{abc} = 0. \quad (3.4)$$

□

As seen in Lemma 3.3 below, the classical limit (correspondence) of quantum theory shows that the variation in the Feynman gauge leads to the classical field equation. In this correspondence of quantum theory to classical theory, the quantum condition by Gupta-Bleuler (or Nakanishi-Lautrup if necessary) in the form of expectation value corresponds to the Lorentz condition. In the Feynman gauge, the action functional, which comprises the Lagrangian density in Eq. (2.12) and an additional term, is expressed as follows:

$$S_{(F)} := \int d^4x \sum_a \left\{ -\frac{1}{4} [F_{\mu\nu}^a(x)]^2 - \frac{1}{2} [\partial_\mu A_\mu^a(x)]^2 \right\}. \quad (3.5)$$

Historically, the variational calculus for an action in analytical dynamics was constructed from differential

equations. It is known that using Eq. (3.3), the above Eq. (3.5) for $A_{(C)\mu}^a$ is reduced to

$$\begin{aligned} S_{(\text{FC})} &:= - \int d^4x \sum_a \frac{1}{2} \partial_\nu A_{(C)\mu}^a \partial_\nu A_{(C)\mu}^a \\ &\quad - \int d^4x \sum_a \frac{1}{2} \partial_\nu (A_{(C)\nu}^a \partial_\mu A_{(C)\mu}^a - A_{(C)\mu}^a \partial_\mu A_{(C)\nu}^a) \\ &= - \int d^4x \sum_a \frac{1}{2} \partial_\nu A_{(C)\mu}^a \partial_\nu A_{(C)\mu}^a. \end{aligned} \quad (3.6)$$

Lemma 3.3. *For the variation of $S_{(\text{FC})}$ with respect to $A_{(C)\mu}^a(x)$, the stationary condition $\delta S_{(\text{FC})} = 0$ leads to the Yang-Mills field equation given by*

$$\partial_\mu \partial_\mu A_{(C)\nu}^a = 0. \quad (3.7)$$

Proof. Using the stationary condition $\delta S_{(\text{FC})} = 0$, Eq. (3.6) leads to Eq. (3.7). \square

It is noted that algebraic properties of non-Abelian Yang-Mills fields without self-interactions are different from those of Abelian fields, because the component of the Yang-Mills field $A^a(x)$ is multiplied by the generator matrix of such as the $\text{SU}(N)$ group to form $A^a(x)T^a$, unlike Abelian fields such as electromagnetic fields.

Subsequently, the author presents a solution $A_{(C)\mu}^a(x)$ to the above differential equation in Eq. (3.7). A field configuration $A_{(\text{CL})\mu}^a(x)$ in Eqs. (3.9)–(3.12) is considered, and the field yields a linear potential between a particle and an antiparticle, because $A_{(\text{CL})\mu}^a(x)$ is multiplied by the group generator T^a . This $A_{(\text{CL})\mu}^a(x)$ does not satisfy the differential equation in Eq. (3.7). By adding $A_{(\text{CU})\mu}^a(x)$ in Eq. (3.28) to $A_{(\text{CL})\mu}^a(x)$, a solution $A_{(C)\mu}^a(x)$ to Eq. (3.7) is derived.

Definition 3.4. The classical field configuration treated in this paper comprises two different functions

$$A_{(C)\mu}^a(x) = A_{(\text{CL})\mu}^a(x) + A_{(\text{CU})\mu}^a(x), \quad (3.8)$$

where $A_{(\text{CL})\mu}^a$ and $A_{(\text{CU})\mu}^a$ are localized and unlocalized fields, respectively.

It is noted that the classical Yang-Mills field is a base field of the quantum fields mentioned below. Physically, the base field is a vacuum, which may be a candidate for a black matter.

The above classical localized function is given in the restricted region. Physically, this is due to the property of a wave packet in the expanding universe with boundaries and the cutoff.

Definition 3.5. For $t, x, y, z \in \mathbb{R}_E^4$, the classical localized function is defined in the region $0 < T_{(b)0} \leq t \leq T_{(b)}$, $|x| \leq X_{(b)}$, $|y| \leq Y_{(b)}$, and $|z| \leq Z_{(b)}$, where the symbols used for the bounds are positive real numbers.

Definition 3.6. In the center-of-mass frame, which is physically used, the localized field $A_{(\text{CL})\mu}^a$ for $1 \leq a \leq 3$ of the group $\text{SU}(2)$ is embedded in the field of the group $\text{SU}(N)$, and is shifted from $A_{(\text{CL})\mu}^a = 0$ to the field

with non-zero components. The classical localized field $A_{(\text{CL})\mu}^a(x)$ are expressed by

$$A_{(\text{CL})t}^a(x) := \lambda^a P_{(0)} s_n^{(0)} \exp(-|c_v^{(0)} x_v|), \quad (3.9)$$

$$A_{(\text{CL})x}^a(x) := 0, \quad (3.10)$$

$$A_{(\text{CL})y}^a(x) := \lambda^a P_{(2)} s_n^{(2)} \exp(-|c_v^{(2)} x_v|), \quad (3.11)$$

$$A_{(\text{CL})z}^a(x) := \lambda^a P_{(3)} s_n^{(3)} \exp(-|c_v^{(3)} x_v|), \quad (3.12)$$

with

$$P_{(0)} := \left(-\frac{1}{2} k_{\text{ND}}^{-1/2} b_v^{(0)} x_v \right) \frac{\exp(-|b_{\mu\nu}^{(\text{M})} x_\mu x_\nu|)}{[1 - \exp(-2|b_{\mu\nu}^{(\text{M})} x_\mu x_\nu|)]^{1/2}}. \quad (3.13)$$

For $a \geq 4$, the components are set as $A_{(\text{CL})\mu}^a = 0$, and λ^a here are defined by constants that depend on the index a for the component of the symmetry group. In the center-of-mass frame, the symbols in Eqs. (3.9) and (3.11)–(3.13) are represented as $b_v^{(0)} = (0, a_c, 0, 0)$ and $c_v^{(0)} = c_v^{(2)} = c_v^{(3)} = (0, 0, 1/d, 1/d)$, where $a_c > 0$ and $d > 0$. The matrix $b_{\mu\nu}^{(\text{M})}$ is written as

$$b_{\mu\nu}^{(\text{M})} := \begin{bmatrix} 0 & \frac{1}{2}a_c & 0 & 0 \\ \frac{1}{2}a_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.14)$$

The variable $s_n^{(0)}$ is a product of the form

$$s_n^{(0)} := s_{nt}^{(0)} s_{nx}^{(0)} s_{ny}^{(0)} s_{nz}^{(0)}, \quad (3.15)$$

and $s_n^{(2)}$ and $s_n^{(3)}$ have the corresponding form, where

$$\begin{aligned} s_{nt}^{(0)} &:= s_{nt}^{(2)} := s_{nt}^{(3)} := +1 && \text{for all } t, \\ s_{nx}^{(0)} &:= s_{nx}^{(2)} := s_{nx}^{(3)} := +1 && \text{for all } x, \\ s_{ny}^{(0)} &:= s_{ny}^{(2)} := s_{ny}^{(3)} := +1 && \text{for } y \geq 0, \\ s_{ny}^{(0)} &:= -s_{ny}^{(2)} := s_{ny}^{(3)} := -1 && \text{for } y < 0, \\ s_{nz}^{(0)} &:= s_{nz}^{(2)} := s_{nz}^{(3)} := +1 && \text{for } z \geq 0, \\ s_{nz}^{(0)} &:= s_{nz}^{(2)} := -s_{nz}^{(3)} := -1 && \text{for } z < 0. \end{aligned} \quad (3.16)$$

The field $A_{(\text{CL})t}^a$ for $x \geq 0$ is then represented by

$$A_{(\text{CL})t}^a(t, \mathbf{x}) = \lambda^a P_{(0)}(t, x) w_t(y) w_t(z), \quad (3.17)$$

where

$$P_{(0)}(t, x) = \frac{1}{2} k_{\text{ND}}^{-1/2} h(t, x), \quad (3.18)$$

with

$$h(t, x) := \frac{-a_c x \exp(-a_c t x)}{[1 - \exp(-2a_c t x)]^{1/2}} \quad \text{for } x \geq \varepsilon_x, \quad (3.19)$$

$$h(t, x) := -h(t, -x) \quad \text{for } x < -\varepsilon_x. \quad (3.20)$$

The symbol ε_x stands for an infinitesimal positive real number, and after the calculations a limit is taken as $\varepsilon_x \rightarrow 0$. The derivative defined is $\partial_x h = \lim_{\Delta x \rightarrow +0} (\Delta h / (\Delta x))$ for $x \geq \varepsilon_x$; and $\partial_x h = \lim_{\Delta x \rightarrow -0} (\Delta h / (\Delta x))$ for $x < -\varepsilon_x$. The function $w_t(y)$ is written as

$$w_t(y) := w(y) := \begin{cases} +\exp(-\frac{y}{d}) & \text{for } y \geq \varepsilon_y \\ -\exp(+\frac{y}{d}) & \text{for } y < -\varepsilon_y \end{cases}, \quad (3.21)$$

where ε_y is an infinitesimal positive real number, and a limit is taken as $\varepsilon_y \rightarrow 0$. The derivative is defined by $\partial_y w = \lim_{\Delta y \rightarrow +0} (\Delta w / (\Delta y))$ for $y \geq \varepsilon_y$; and $\partial_y w = \lim_{\Delta y \rightarrow -0} (\Delta w / (\Delta y))$ for $y < -\varepsilon_y$. The forms for z are the same as those for y .

The fields $A_{(\text{CL})y}^a(t, \mathbf{x})$ and $A_{(\text{CL})z}^a(t, \mathbf{x})$ are written as

$$A_{(\text{CL})y}^a(t, \mathbf{x}) = \lambda^a \left(\frac{d}{2} \right) (\partial_t P_{(0)}) w_y(y) w(z), \quad (3.22)$$

$$A_{(\text{CL})z}^a(t, \mathbf{x}) = \lambda^a \left(\frac{d}{2} \right) (\partial_t P_{(0)}) w(y) w_z(z), \quad (3.23)$$

where

$$w_y(y) := \begin{cases} +w(y) & \text{for } y \geq \varepsilon_y \\ -w(y) & \text{for } y < -\varepsilon_y \end{cases}, \quad (3.24)$$

$$w_z(z) := \begin{cases} +w(z) & \text{for } z \geq \varepsilon_z \\ -w(z) & \text{for } z < -\varepsilon_z \end{cases}. \quad (3.25)$$

Lemma 3.7. *Let $A_{(\text{CL})\mu}^a(x)$ be the classical localized functions of the Yang-Mills field. Then $A_{(\text{CL})\mu}^a(x)$ satisfies the following Lorentz condition*

$$\partial_\mu A_{(\text{CL})\mu}^a(x) = 0, \quad (3.26)$$

where the derivatives are defined in the region in which the function is defined.

Proof. With the help of Definition 3.6 involving Eqs. (3.10), (3.17), and (3.21)–(3.25), it follows that

$$\begin{aligned} \partial_\mu A_{(\text{CL})\mu}^a(x) &= \lambda^a [(\partial_t P_{(0)}) w(y) w(z) + 0 \\ &\quad - \frac{1}{2} (\partial_t P_{(0)}) w(y) w(z) - \frac{1}{2} (\partial_t P_{(0)}) w(y) w(z)] = 0. \end{aligned} \quad (3.27)$$

□

The localized function $A_{(\text{CL})\text{v}}^a(t, \mathbf{x})$ satisfies Eq. (3.31), where $\rho_{\text{v}}^a(t, \mathbf{x})$ is given by such as Eq. (3.32). The unlocalized function $A_{(\text{CU})\text{v}}^a(t, \mathbf{x})$, which is different from $A_{(\text{CL})\text{v}}^a(t, \mathbf{x})$, as shown in Eq. (3.38), is introduced in Eq. (3.28) to satisfy Eq. (3.35). Then $A_{(\text{CL})\text{v}}^a(t, \mathbf{x}) + A_{(\text{CU})\text{v}}^a(t, \mathbf{x})$ becomes a solution to Eq. (3.37).

Definition 3.8. For $x \in \mathbb{R}_{\text{E}}^4$, the unlocalized function in Eq. (3.8) has the form

$$A_{(\text{CU})\mu}^a(t, \mathbf{x}) := \int_{T_{(\text{b})}}^t dt_s \int_{-X_{(\text{b})}}^{X_{(\text{b})}} dx_s \int_{-Y_{(\text{b})}}^{Y_{(\text{b})}} dy_s \int_{-Z_{(\text{b})}}^{Z_{(\text{b})}} dz_s \times G_4(t, \mathbf{x}; t_s, \mathbf{x}_s) \rho_{(\text{CU})\mu}^a(t_s, \mathbf{x}_s), \quad (3.28)$$

where G_4 is the Green function denoted by

$$G_4(t, \mathbf{x}; t_s, \mathbf{x}_s) := \frac{1}{2\pi^2} \frac{1}{(t-t_s)^2 + (x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2}, \quad (3.29)$$

which satisfies

$$\partial_\mu^2 A_{(\text{CU})\text{v}}^a(t, \mathbf{x}) = -\rho_{(\text{CU})\text{v}}^a(t, \mathbf{x}), \quad (3.30)$$

with $\rho_{(\text{CU})}$ being four-dimensional (4D) charge-current density of the classical unlocalized field. In the denominator in Eq. (3.29), an infinitesimal positive real number ϵ_{Gr} is added to $(t-t_s)^2 + (x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2$, and after the calculations the limit is taken as $\epsilon_{\text{Gr}} \rightarrow 0$. If necessary, the Green function is extended.

Lemma 3.9. The classical localized function $A_{(\text{CL})\text{v}}^a(t, \mathbf{x})$ satisfies

$$\partial_\mu^2 A_{(\text{CL})\text{v}}^a(t, \mathbf{x}) = -\rho_{(\text{CL})\text{v}}^a(t, \mathbf{x}) = -\rho_{\text{v}}^a(t, \mathbf{x}), \quad (3.31)$$

where $\rho_{(\text{CL})\text{v}}^a(t, \mathbf{x}) = \rho_{\text{v}}^a(t, \mathbf{x})$ is the 4D charge-current density of the classical localized function $A_{(\text{CL})\mu}^a(x)$ in Eq. (3.8), and is derived such as

$$\rho_t^a(t, x, y, z) = \begin{cases} \lambda^a \left[\left(\frac{1}{2} k_{\text{Nb}}^{-1/2} \right) Q(t, x) - \frac{2}{d^2} P_{(0)}(t, x) \right] w(y)w(z) & \text{for } \epsilon_x \leq x \leq X_c \\ 0 & \text{for } x > X_c \end{cases}, \quad (3.32)$$

where X_c is the wave-packet size of the classical localized function, and

$$Q(t, x) := a_c^3 x^3 Q_1(t, x) - 2a_c^2 t Q_2(t, x) + a_c^3 t^2 x Q_1(t, x), \quad (3.33)$$

with

$$\begin{aligned} Q_1(t, x) &:= Q_{1/2}(t, x) + 4Q_{3/2}(t, x) + 3Q_{5/2}(t, x), \\ Q_2(t, x) &:= Q_{1/2}(t, x) + Q_{3/2}(t, x), \\ Q_{1/2}(t, x) &:= \frac{\exp(-a_c |tx|)}{[1 - \exp(-2a_c |tx|)]^{1/2}}, \\ Q_{3/2}(t, x) &:= \frac{\exp(-3a_c |tx|)}{[1 - \exp(-2a_c |tx|)]^{3/2}}, \\ Q_{5/2}(t, x) &:= \frac{\exp(-5a_c |tx|)}{[1 - \exp(-2a_c |tx|)]^{5/2}}. \end{aligned} \quad (3.34)$$

Proof. This lemma is obtained straightforward from Definition 3.6. \square

Lemma 3.10. *By setting*

$$\rho_{(\text{CU})\nu}^a(t, \mathbf{x}) = -\rho_{\nu}^a(t, \mathbf{x}), \quad (3.35)$$

for the current density of the unlocalized field, the classical Yang-Mills field $A_{(\text{C})\nu}^a(x)$ in Eq. (3.8) satisfies

$$\partial_{\mu}^2 A_{(\text{C})\nu}^a(x) = 0. \quad (3.36)$$

Proof. Using Eqs. (3.30) and (3.31),

$$\begin{aligned} & \partial_{\mu}^2 (A_{(\text{CL})\nu}^a(x) + A_{(\text{CU})\nu}^a(x)) \\ &= -\rho_{(\text{CL})\nu}^a(t, \mathbf{x}) - \rho_{(\text{CU})\nu}^a(t, \mathbf{x}) \\ &= -\rho_{\nu}^a(t, \mathbf{x}) + \rho_{\nu}^a(t, \mathbf{x}) = 0. \end{aligned} \quad (3.37)$$

\square

Lemma 3.11. *The unlocalized field $A_{(\text{CU})\nu}^a(t, \mathbf{x})$ differs from the localized field $A_{(\text{CL})\nu}^a(t, \mathbf{x})$, namely,*

$$A_{(\text{CU})\nu}^a(t, \mathbf{x}) \neq A_{(\text{CL})\nu}^a(t, \mathbf{x}), \quad (3.38)$$

which states that Eq. (3.36) is meaningful as the Yang-Mills field.

Proof. From Eqs. (3.21) and (3.32), it follows that

$$\rho_t^a(t_s, x_s, -y_s, z_s) = -\rho_t^a(t_s, x_s, y_s, z_s), \quad (3.39)$$

$$\rho_t^a(t_s, x_s, y_s, -z_s) = -\rho_t^a(t_s, x_s, y_s, z_s), \quad (3.40)$$

canceling the charge contributions to the integral for the Green function in Eq. (3.28) at $y = 0$ and $z = 0$, and then the unlocalized field takes the value

$$A_{(\text{CU})t}^a(t, \mathbf{x}) = 0 \quad \text{at } y = 0 \text{ and } z = 0. \quad (3.41)$$

By contrast, the localized function $A_{(\text{CL})t}^a(t, \mathbf{x})$ does not always take the value of zero at $y = 0$ and $z = 0$, as can be seen, with the help of Eqs. (3.17)–(3.21). \square

3.2.2 Classical Wilson loop

In this subsection, a potential between a particle and an antiparticle for the stationary classical field is considered using the Wilson loop. These paired particles exist in the pure Yang-Mills field, and have opposite color charges, which imply the charges of the $\text{SU}(N)$ symmetry. The Wilson loop is defined in Eq. (3.45) as trace of an exponential function of a closed line integral of the field. The trace obtains a physical quantity, and the potential in Eq. (3.66) is derived from the logarithmic (inverse) function of the exponential function.

Notably, properties of the group are used in such as Eq. (3.64), because the field component is multiplied by the generator of the group. Before presenting the Wilson loop for the Yang-Mills field, an example of the loop integral in the case of an Abelian field is briefly described. For $x \in \mathbb{R}_E^4$, let's consider the following integral

$$\begin{aligned} I_{(\text{CA})} &:= \oint dx_\mu A_\mu(x) = \oint dx_\mu A_\mu(t, x, y, z) \\ &= I_{(\text{CA})}^{(1)} + I_{(\text{CA})}^{(2)} + I_{(\text{CA})}^{(3)} + I_{(\text{CA})}^{(4)}, \end{aligned} \quad (3.42)$$

where

$$\begin{aligned} I_{(\text{CA})}^{(1)} &:= \int_{x_1}^{x_2} dx gA_x(t_1, x, 0, 0), \\ I_{(\text{CA})}^{(2)} &:= \int_{t_1}^{t_2} dt gA_t(t, x_2, 0, 0), \\ I_{(\text{CA})}^{(3)} &:= \int_{x_2}^{x_1} dt gA_x(t_2, x, 0, 0), \\ I_{(\text{CA})}^{(4)} &:= \int_{t_2}^{t_1} dt gA_t(t, x_1, 0, 0). \end{aligned} \quad (3.43)$$

Here, a unit charge and a unit anticharge are in interaction with the field in the above $I_{(\text{CA})}^{(2)}$ and $I_{(\text{CA})}^{(4)}$, respectively, for a coupling constant g . By setting $A_x = 0$ in the nearly static case, it follows that

$$I_{(\text{CA})} = g[A_t(t_1, x_2, 0, 0) - A_t(t_1, x_1, 0, 0)](t_2 - t_1), \quad (3.44)$$

which implies the potential depends on the distance between the charges. This loop integral is generalized to $\text{Tr}[\exp(-iI_{(\text{CA})})]$, which is available to both Abelian and non-Abelian fields.

Definition 3.12. The Wilson loop of the classical pure Yang-Mills field $A_{(\text{C})\mu}(x)$ is defined by the form

$$W_C := \text{Tr}[\exp(-ig \oint dx_\mu A_{(\text{C})\mu}(x))]. \quad (3.45)$$

To obtain the Wilson loop, the following line integrals are performed along a closed rectangular loop whose sides are parallel to the x - or t -axis

$$\begin{aligned} I_{(\text{CL})}^{a(1)} &:= \int_{x_1}^{x_2} dx A_{(\text{CL})x}^a(t_1, x, 0, 0), \\ I_{(\text{CL})}^{a(2)} &:= \int_{t_1}^{t_2} dt A_{(\text{CL})t}^a(t, x_2, 0, 0), \\ I_{(\text{CL})}^{a(3)} &:= \int_{x_2}^{x_1} dx A_{(\text{CL})x}^a(t_2, x, 0, 0), \\ I_{(\text{CL})}^{a(4)} &:= \int_{t_2}^{t_1} dt A_{(\text{CL})t}^a(t, x_1, 0, 0), \end{aligned} \quad (3.46)$$

where $x_2 = -x_1$.

Lemma 3.13. *The line integrals contained in the Wilson loop for the classical unlocalized field vanish.*

Proof. The x -component of the unlocalized field satisfies $A_{(\text{CU})x}^a(t, \mathbf{x}) = 0$ from Eqs. (3.10), (3.28), (3.31), and (3.35), then the line integrals along the x -axis vanish as written by

$$\begin{aligned} I_{(\text{CU})}^{a(1)} &:= \int_{x_1}^{x_2} dx A_{(\text{CU})x}^a(t_1, x, 0, 0) = 0, \\ I_{(\text{CU})}^{a(3)} &:= \int_{x_2}^{x_1} dx A_{(\text{CU})x}^a(t_2, x, 0, 0) = 0. \end{aligned} \quad (3.47)$$

In addition, with the aid of Eq. (3.41), the following relations are derived

$$\begin{aligned} I_{(\text{CU})}^{a(2)} &:= \int_{t_1}^{t_2} dx A_{(\text{CU})t}^a(t, x_1, 0, 0) = 0, \\ I_{(\text{CU})}^{a(4)} &:= \int_{x_1}^{x_2} dx A_{(\text{CU})t}^a(t, x_2, 0, 0) = 0. \end{aligned} \quad (3.48)$$

From Eqs. (3.47) and (3.48), the sum of the contributions of the classical unlocalized field to the Wilson loop thus vanishes as

$$I_{(\text{CU})}^a := I_{(\text{CU})}^{a(1)} + I_{(\text{CU})}^{a(2)} + I_{(\text{CU})}^{a(3)} + I_{(\text{CU})}^{a(4)} = 0. \quad (3.49)$$

□

Lemma 3.14. *The total line integral $I_{(\text{CL})}^a$ in the Wilson loop for the classical localized field has the form*

$$I_{(\text{CL})}^a = -\lambda^a (k_{\text{ND}}^{-1/2}) \arccos[\exp(-a_c x_2 t_2)]. \quad (3.50)$$

Proof. From Eq. (3.10), the line integrals along the x -axis in Eq. (3.46) yield

$$I_{(\text{CL})}^{a(1)} = 0, \quad I_{(\text{CL})}^{a(3)} = 0. \quad (3.51)$$

Whereas, using Definition 3.6 that contains Eqs. (3.17)–(3.21), the integrals along the t -axis in Eq. (3.46) provide

$$I_{(\text{CL})}^{a(2)} + I_{(\text{CL})}^{a(4)} = \frac{\lambda^a k_{\text{ND}}^{-1/2}}{2} [H_{(\text{CL})}(t_2, x_2) - H_{(\text{CL})}(t_1, x_2)], \quad (3.52)$$

where $x_2 = -x_1$. From Eqs. (3.19) and (3.20), it follows that

$$H_{(\text{CL})}(t, x) := \int dt' h(t', x) = -\arccos[\exp(-a_c x t)], \quad (3.53)$$

where $H(t, x)$ has the symmetry $H(t, x) = -H(t, -x)$, and summing the above integrals results in

$$\begin{aligned} I_{(\text{CL})}^a &= I_{(\text{CL})}^{a(1)} + I_{(\text{CL})}^{a(2)} + I_{(\text{CL})}^{a(3)} + I_{(\text{CL})}^{a(4)} \\ &= -\lambda^a k_{\text{ND}}^{-1/2} \{ \arccos[\exp(-a_c x_2 t_2)] \\ &\quad - \arccos[\exp(-a_c x_2 t_1)] \}. \end{aligned} \quad (3.54)$$

By neglecting the last term $\arccos[\exp(-a_c x_2 t_1)]$ for small $a_c x_2 t_1$, Eq. (3.50) is thus derived. □

Lemma 3.15. *Only the classical localized field contributes to the total line integral $I_{(C)}^a$ in the Wilson loop, as written by*

$$I_{(C)}^a = I_{(CL)}^a. \quad (3.55)$$

Proof. With the aid of Eqs. (3.49) and (3.54), the Wilson loop is reduced to the following integral to give the above lemma

$$I_{(C)}^a = I_{(CL)}^a + I_{(CU)}^a = I_{(CL)}^a. \quad (3.56)$$

□

Lemma 3.16. *The classical Wilson loop yields a classical potential V_C between a charge and an anticharge of the Yang-Mills field of the form*

$$V_C = \frac{a_c}{2}(x_2 - x_1). \quad (3.57)$$

Proof. Using Eq. (3.50), the classical Wilson loop in Eq. (3.45) becomes

$$W_C = \text{Tr}[\exp(i \sum_a y' g \lambda^a k_{N_D}^{-1/2} T^a)], \quad (3.58)$$

with

$$y' := \arccos[\exp(-a_c x_2 t_2)]]. \quad (3.59)$$

The value of the normalization constant k_{N_D} for $\lambda^1 = \lambda^2 = \lambda^3$ is given to satisfy

$$\begin{aligned} & k_{N_D}^{-1} g_c^2 \sum_{a=1}^3 \sum_{b=1}^3 \lambda^a \lambda^b T^a T^b \\ &= \sum_{a=1}^3 \sum_{b=1}^3 (k_{N_D}^{-1/2} g_c \lambda^a T^a) (k_{N_D}^{-1/2} g_c \lambda^b T^b) \\ &= I^{(3)}, \end{aligned} \quad (3.60)$$

where $I^{(3)}$ is the unit matrix in the form of 2×2 submatrices, such as $SU(2)$, embedded in the $N \times N$ matrix, which has individual elements equal to zero. For the quantity in the above equation, ξ^a is defined by

$$\xi^a := k_{N_D}^{-1/2} g_c \lambda^a. \quad (3.61)$$

Then, the following series are considered

$$\begin{aligned} \cos(y) &= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}, \\ \sin(y) &= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}. \end{aligned} \quad (3.62)$$

With the help of the properties of the matrices T^a , representing generators of the SU(2) group,

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta_{ab}, \quad \text{Tr}(T^a) = 0, \quad (3.63)$$

$\text{Tr}[\exp(iy' \sum_{a=1}^3 \xi^a T^a)]$ is simplified as

$$\begin{aligned} & \text{Tr}[\exp(iy' \sum_{a=1}^3 \xi^a T^a)] \\ &= \text{Tr}[\cos(y' \sum_{a=1}^3 \xi^a T^a)] + i \text{Tr}[\sin(y' \sum_{a=1}^3 \xi^a T^a)] \\ &= \text{Tr}[\cos(y' I^{(3)})] = [\text{Tr}(I^{(3)})] \cos(y'). \end{aligned} \quad (3.64)$$

Then, using Eq. (3.58), the classical Wilson loop, which is obtained as a sum of the integrals in Eq. (3.46) for small $a_c x_2 t_1$, becomes

$$\begin{aligned} W_C &= [\text{Tr}(I^{(3)})] \left\{ \cos\{\arccos[\exp(-a_c x_2 t_2)]\} \right\} \\ &= [\text{Tr}(I^{(3)})] \exp(-a_c x_2 t_2) \\ &= \exp[-a_c x_2 (t_2 - t_1) + \ln[\text{Tr}(I^{(3)})]] \\ &= \exp\left[-\frac{a_c}{2} (x_2 - x_1)(t_2 - t_1) + \ln[\text{Tr}(I^{(3)})]\right], \end{aligned} \quad (3.65)$$

where the relation $x_2 = -x_1$ has been used. It then follows that

$$V_C = -\frac{\ln(W_C)}{t_2 - t_1} = \frac{a_c}{2} (x_2 - x_1), \quad (3.66)$$

where the last term in Eq. (3.65) was neglected. \square

It is note that that this linear potential is not derived for Abelian fields, since the Abelian fields do not involve the matrices T^a of the group.

3.2.3 Scale-invariant energy of Yang-Mills fields, string tension of the linear potential, and Polyakov's binding energy

The Yang-Mills field contains a basic constant that is not determined theoretically. This constant is a given scale-invariant energy that indicates the size of a pair composed of a particle and an antiparticle. In the case of an atom, this size corresponds to an atomic size, which is determined using the fundamental constants such as the Planck constant and electronic charge. The liner string tension is related to the above scale-invariant energy. Furthermore, the binding energy between a particle and an antiparticle is derived for the pure Yang-Mills field. In these analysis processes, the energy of the Yang-Mills field is calculated using Eq. (3.68), and Eq. (3.73) is obtained. The decrease/increase in the energy of the Yang-Mills field is equal to the increase/decrease in the energy of the paired particles through the linear potential. This equality provides the relationship between the slope of the linear potential and the distance between the paired particles, as

indicated in Eq. (3.76). The binding energy, which is the mass of the pair of the particles, is calculated using the Polyakov line in Eq. (3.78), yielding Eq. (3.80).

Physically, an atom has the characteristic Bohr radius, which is expressed in terms of Planck's constant, electron mass, and coupling constant in the Abelian field theory. By contrast, it is known that non-Abelian Yang-Mills fields have a somewhat different intrinsic scale-invariant energy, which is equivalently length or time. This fundamental scale-invariant energy is a given constant similar to such as the Planck's constant, and cannot be determined from theory. It is therefore required that a solution to the Yang-Mills equation be consistent with this scale-invariance. Here, the present theory introduces the scale-invariant energy of the classical field as a vacuum at Minkowski time t' , corresponding to Euclidean time t .

Definition 3.17. Let $\varepsilon_{t'}$ be a positive real number in Minkowski time t' . Then the existence of classical fields is limited to $|x| \leq \varepsilon_{t'}$, where $c = 1$ with the speed of light c .

Lemma 3.18. The energy of the classical localized field $E_{(\text{CL})}(\varepsilon_{t'})$ at the Minkowski time t' is given by

$$E_{(\text{CL})}(\varepsilon_{t'}) = \frac{\zeta(3)\Gamma(3)}{4g_c^2} \frac{1}{a_c(\varepsilon_{t'})^3}, \quad (3.67)$$

where ζ and Γ are zeta and gamma functions, respectively.

Proof. The energy of the classical localized field $E_{(\text{CL})}(\varepsilon_{t'})$ is expressed with the aid of the energy-momentum tensor [28] $T_{00(\text{CL})}$

$$E_{(\text{CL})} = \int dx dy dz \ T_{00(\text{CL})}. \quad (3.68)$$

Here

$$T_{00(\text{CL})} = \frac{\partial L_{\text{F}(\text{CL})}}{\partial(\frac{\partial A_{(\text{CL})\nu}}{\partial x'_0})} \frac{\partial A_{(\text{CL})\nu}}{\partial x'_0} - L_{\text{F}(\text{CL})}, \quad (3.69)$$

where

$$L_{\text{F}(\text{CL})} := -\frac{2}{4} \text{Tr}(F_{(\text{CL})\mu\nu})^2. \quad (3.70)$$

Then let us use the aforementioned field strength $F_{(\text{CL})\mu\nu}$ at time t' in the limit as $d \rightarrow 0$ with d being the thickness of the classical localized field. The dominant term is the integral of squared field strengths $[-\partial_y A_{(\text{CL})t'}^a]^2$ in $[F_{(\text{CL})yt'}^a]^2$ and $[-\partial_z A_{(\text{CL})t'}^a]^2$ in $[F_{(\text{CL})zt'}^a]^2$, leading to

$$\begin{aligned} E_{(\text{CL})}(\varepsilon_{t'}) &= \{2 \int_0^\infty dx \ [\frac{1}{g_c} h(\varepsilon_{t'}, x)]^2\} \\ &\times \frac{4}{2} \{ \int_0^\infty \int_0^\infty dy dz \ [(\partial_y w(y)w(z))^2 + (w(y)\partial_z w(z))^2] \}. \end{aligned} \quad (3.71)$$

This integral does not depend on d , and limits are taken as $\varepsilon_y \rightarrow 0$, $\varepsilon_z \rightarrow 0$, and $\varepsilon_x \rightarrow 0$. Using Eqs. (3.19)–(3.21) and

$$\int_0^\infty dx (h(\varepsilon_{t'}, x))^2 = \zeta(3)\Gamma(3)a_c^2 \frac{1}{(2a_c\varepsilon_{t'})^3}, \quad (3.72)$$

the integral in (3.71) results in

$$E_{(\text{CL})}(\epsilon_{t'}) = \frac{\zeta(3)\Gamma(3)}{4g_c^2} \frac{1}{(\epsilon_{t'})^3 a_c}. \quad (3.73)$$

□

Lemma 3.19. *For charges of gluons, namely, particles of the Yang Mills field, let's consider a pair of a charge and an anticharge, which are separated by the distance $2\epsilon_{t'}$ with $c = 1$, where c is physically the speed of light. Let $E_{(\text{CL})}(\epsilon_{t'})$ be the energy of the classical localized field, and let $E_{(\text{LP})}(\epsilon_{t'}) = |-\frac{a_c}{2}(2\epsilon_{t'})| = |-\sigma(2\epsilon_{t'})|$ be the stabilized/unstabilized energy of the pair. The following relation can be set*

$$E_{(\text{CL})}(\epsilon_{t'}) = E_{(\text{LP})}(\epsilon_{t'}), \quad (3.74)$$

$$\frac{1}{\lambda_{\text{MOM}}} = 4\epsilon_{t'}, \quad (3.75)$$

with λ_{MOM} being the intrinsic scale-invariant energy. This λ_{MOM} is not determined from theory. Then, the absolute string tension (energy per unit length) σ has the form

$$\sigma = \frac{1}{2}a_c = \frac{4\zeta(3)^{1/2}\Gamma(3)^{1/2}\lambda_{\text{MOM}}^2}{g_c}. \quad (3.76)$$

Proof. With the use of Eqs. (3.73)–(3.75), the above Eq.(3.76) is obtained. □

The binding energy of a charge-anticharge pair is obtained from the Polyakov line [29] in a similar way to the Wilson loop.

Definition 3.20. Corresponding to physics, let k_B and T be the Boltzmann constant and temperature, respectively. Then τ is defined by $\tau = 1/k_B T$. By replacing t by τ' and under the periodic condition along the temperature axis, the classical localized function at finite T has the following form

$$\begin{aligned} A_{(\text{CL})t}^a(t, x, y, z) &\rightarrow A_{(\text{CL})\tau}^a(\tau', x, y, z) \\ &= A_{(\text{CL})t}^a(\tau', x, y, z) + A_{(\text{CL})t}^a(\tau - \tau', x, y, z). \end{aligned} \quad (3.77)$$

The Polyakov line is defined by the line integral

$$P_\tau := \text{Tr}[\exp(-ig_c \sum_a I_P^a T^a)], \quad (3.78)$$

with

$$I_P^a := \int_{\tau_\epsilon}^{\tau - \tau_\epsilon} d\tau' A_{(\text{CL})\tau}^a(\tau', x_2, 0, 0), \quad (3.79)$$

where the starting point returns to the end point, and τ_ϵ is an infinitesimally small real number. It is known that the Polyakov line yields the following binding energy of a charge-anticharge pair

$$\epsilon_q = -\ln(P_\tau). \quad (3.80)$$

Lemma 3.21. *For the charge-anticharge distance r , the Polyakov line gives the following binding energy*

$$\varepsilon_q = \frac{\sigma r}{k_B T}. \quad (3.81)$$

This equation states that for large τ , physically at low temperatures, the non-zero binding energy leads to a finite mass gap of the pure Yang-Mills field due to the pairing of a charge and an anticharge.

Proof. With the aid of Eqs. (3.54) and (3.65), the Polyakov line is represented by

$$P_\tau = \cos\{\arccos[\exp(-\sigma r \tau)] - \arccos[\exp(-\sigma r \tau_\varepsilon)]\}, \quad (3.82)$$

where constant was neglected. When $\tau_\varepsilon \rightarrow 0$, this P_τ becomes

$$P_\tau = \cos\{\arccos[\exp(-\sigma r \tau)]\} = \exp(-\sigma r \tau). \quad (3.83)$$

Denoting E_B by $E_B = \sigma r$, the binding energy is thus derived as

$$\begin{aligned} \varepsilon_q &= -\ln(P_\tau) = -\ln\left[\exp\left(\frac{-E_B}{k_B T}\right)\right] \\ &= \frac{E_B}{k_B T} = \frac{\sigma r}{k_B T}. \end{aligned} \quad (3.84)$$

□

3.3 Quantum field in path integral around the classical field as a vacuum

3.3.1 Quantum action of pure Yang-Mills fields expanded in terms of scalar plane wave functions

Here, the additional potential between a particle and an antiparticle is derived considering quantum fluctuations from the aforementioned stationary state. The Wilson loop for quantum fluctuations is calculated using the Feynman path integral. For the sufficiently large cutoff value, the coupling constant is small, and the action functional in the Feynman gauge is given by Eq. (3.87). By expanding the quantum field in terms of the plane-wave basis functions in Eq. (3.91), the path integral is reduced to the Gauss integral in Eq. (3.107). The potential is independent of the generators of the group in Eq. (3.111), and has the same form as that of the U(1) group. The potential reveals to be a Coulomb potential, and the absolute Coulomb potential is larger than the absolute linear potential at short distances. As aforementioned, the pure Yang-Mills field $A_\mu^a(x)$ in the present formalism is expressed as

$$A_\mu^a(x) = A_{(C)\mu}^a(x) + A_{(Q)\mu}^a(x), \quad (3.85)$$

where $A_{(C)\mu}(x)$ is the classical field as a vacuum and $A_{(Q)\mu}^a(x)$ is the quantum field, which is the fluctuation around the classical field. When necessary, Faddeev-Popov (FP) ghost is added. Then, the zeroth order of the action for the non-Abelian Yang-Mills fields is the classical field, and the first-order action vanishes due to the classical equations of motion, while the second order remains.

Definition 3.22. Let's consider the following quantum action with the coupling constant g in the Feynman gauge

$$S = S^{(2)} + S^{(3)} + S^{(4)}, \quad (3.86)$$

where

$$S^{(2)} := -\frac{1}{2} \sum_a \int d^4x (\partial_\nu A_{(Q)\mu}^a \partial_\nu A_{(Q)\mu}^a), \quad (3.87)$$

$$S^{(3)} := g \sum_{a,b,c} f^{abc} \int d^4x (A_{(Q)\mu}^b A_{(Q)\nu}^c \partial_\mu A_{(C)\nu}^a + A_{(Q)\mu}^b A_{(C)\nu}^c \partial_\mu A_{(Q)\nu}^a + A_{(C)\mu}^b A_{(Q)\nu}^c \partial_\mu A_{(Q)\nu}^a + A_{(Q)\mu}^b A_{(Q)\nu}^c \partial_\mu A_{(Q)\nu}^a), \quad (3.88)$$

$$S^{(4)} := -\frac{g^2}{4} \sum_{a,b,c,d,e} f^{abc} f^{ade} \int d^4x (A_{(Q)\mu}^b A_{(Q)\nu}^c A_{(C)\mu}^d A_{(C)\nu}^e + A_{(Q)\mu}^b A_{(C)\nu}^c A_{(Q)\mu}^d A_{(C)\nu}^e + A_{(C)\mu}^b A_{(Q)\nu}^c A_{(Q)\mu}^d A_{(C)\nu}^e + A_{(Q)\mu}^b A_{(C)\nu}^c A_{(C)\mu}^d A_{(Q)\nu}^e + A_{(C)\mu}^b A_{(Q)\nu}^c A_{(C)\mu}^d A_{(Q)\nu}^e + A_{(Q)\mu}^b A_{(C)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e + A_{(C)\mu}^b A_{(Q)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e + A_{(Q)\mu}^b A_{(Q)\nu}^c A_{(C)\mu}^d A_{(Q)\nu}^e + A_{(Q)\mu}^b A_{(C)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e + A_{(C)\mu}^b A_{(Q)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e + A_{(Q)\mu}^b A_{(C)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e + A_{(C)\mu}^b A_{(Q)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e + A_{(Q)\mu}^b A_{(C)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e + A_{(C)\mu}^b A_{(Q)\nu}^c A_{(Q)\mu}^d A_{(Q)\nu}^e). \quad (3.89)$$

The above quantum action comprises kinetic and self-interaction terms, and the latter in the cubic and quartic form almost contain the classical field. The cubic term is small because the quantum coupling constant g is small for the cutoff at the high energy due to the asymptotic freedom; and the quartic term is smaller than the cubic term for such the cutoff at the high energy.

The contributions of the quantum fluctuations are analyzed using the path integral.

Definition 3.23. Using Definition 2.5, the quantum pure Yang-Mills fields are expanded in terms of basis functions $\chi_k(x)$ within the cutoff energy as

$$A_{(Q)\mu}^a(x) = \sum_k A_{(Q)\mu k}^a \chi_k(x), \quad (3.90)$$

where considering Definition 2.5, the basis functions $\chi_k(x)$ are denoted by

$$\chi_k(x) = \exp(ik_\mu x_\mu) \quad \text{with } |k| \leq \Lambda_{\text{CO}}, \quad (3.91)$$

Since the energy of the localized field is independent of d in Eq. (3.21), one can consider the case in which a limit is taken as $d \rightarrow 0$ for the classical field. Before the lemma below, let us consider a scalar field $\phi_M(x)$ with a mass m_M , which was introduced by Klein and Gordon, and was used by Yukawa. The Lagrangian density of the mass term for this field has the form

$$\mathcal{L}_M \propto \frac{1}{2} \phi_M(x) (m_M)^2 \phi_M(x). \quad (3.92)$$

Lemma 3.24. Let $S_{(\text{CQ})}^{(41)}$ be an integral denoted by

$$S_{(\text{CQ})}^{(41)} := -\frac{1}{4}g^2 f^{abc} f^{ade} \int d^4x A_{(\text{Q})\mu}^b(x) A_{(\text{C})t}^c(x) A_{(\text{C})t}^e(x) A_{(\text{Q})\mu}^d(x). \quad (3.93)$$

For $b = d$ and $c = e$, it follows that

$$S_{(\text{CQ})}^{(41)} = -\frac{1}{4}g^2 (f^{abc})^2 \int d^4x A_{(\text{Q})\mu}^b(x) [A_{(\text{C})t}^c(x)]^2 A_{(\text{Q})\mu}^b(x). \quad (3.94)$$

As aforementioned, the summation notation is employed solely for the indices referring to spacetime coordinates. In the case of $A_{(\text{C})t}^c(x) \neq 0$, from the above equation, the relation,

$$m_{\text{loc}} = \frac{1}{4}g^2 (f^{abc})^2 [A_{(\text{C})t}^c(x)]^2 > 0, \quad (3.95)$$

states that the pure Yang-Mill field has a local mass gap of $(2m_{\text{loc}})^{1/2}$.

Proof. In Eq. (3.95), squared non-zero structure constants take the positive value as written by

$$(f^{abc})^2 > 0. \quad (3.96)$$

Hence, m_{loc} and $(2m_{\text{loc}})^{1/2}$ have each positive value, with the aid of Eq. (3.92). \square

3.3.2 Quantum Wilson loop

Here, the dominant quantum fluctuations to the Wilson loop is presented, considering that higher-order terms do not contribute to the loop, due to the asymptotic freedom having small coupling for the cutoff at the high energy via renormalizations.

Definition 3.25. Let's denote the action of the kinetic terms in Eq. (3.87) by $S_{\text{Q,Q}}^{(2)a}$. Then this term is expanded in terms of the basis functions to give

$$\begin{aligned} S_{\text{Q,Q}}^{(2)a} &:= -\frac{1}{2} \int d^4x [\partial_\nu \sum_k A_{(\text{Q})\mu k}^a \chi_k(x)] [\partial_\nu \sum_{k'} A_{(\text{Q})\mu k'}^a \chi_{k'}(x)] \\ &= \sum_{k,k'} A_{(\text{Q})\mu k}^a M_{kk'}^{\text{Q,Q}} A_{(\text{Q})\mu k'}^a, \end{aligned} \quad (3.97)$$

where

$$M_{kk'}^{\text{Q,Q}} := -\frac{1}{2} \int d^4x [\partial_\nu \chi_k(x)] [\partial_\nu \chi_{k'}(x)], \quad (3.98)$$

with each diagonal eigenvalue $\eta_{(\mu)k}$ in the form proportional to k^2 for the plane-wave basis functions.

Definition 3.26. The quantum action S_{Q} considered for the above $S_{\text{Q,Q}}^{(2)a}$ is expressed by

$$S_{\text{Q}} = \sum_a \sum_{k,k'} M_{kk'}^{\text{Q,Q}} A_{(\text{Q})\mu k}^a A_{(\text{Q})\mu k'}^a. \quad (3.99)$$

By using the classical Wilson loop W_C , in which the 2×2 unit submatrix $I^{(3)}$ embedded in the $N \times N$ matrix with vanishing components, the Wilson loop W_Q in the quantum path integral has the form

$$W_Q = \text{Tr} \left\{ W_C I^{(3)} \frac{1}{Z_N} \int D[A_{(Q)\mu k}^a] \exp(-S_Q) \exp(C) \right\}. \quad (3.100)$$

Here, C is denoted by

$$C := -ig \sum_a \sum_k \beta_{\mu k} A_{(Q)\mu k}^a T^a, \quad (3.101)$$

where

$$\beta_{\mu k} := \oint dx_\mu \chi_k(x). \quad (3.102)$$

The quantity Z_N is a normalization constant in the path integral with respect to $A_{(Q)\mu k}^a$, and is written as

$$Z_N := \int D[A_{(Q)\mu k}^a] \exp(-S_Q). \quad (3.103)$$

In Eq. (3.100), the contribution from the classical field gives rise to the prefactor. Meanwhile, the quantum fluctuations $A_{(Q)\mu k}^a$ of the pure Yang-Mills field such as the SU(3) gauge field are represented in terms of the matrices T^a that represent the SU(3) group as an example.

Lemma 3.27. *The analytical confining potential $V(x_2 - x_1)$ between a Yang-Mills charge and an anticharge separated by the distance $x_2 - x_1$ is composed of the first classical linear term plus the second quantum Coulomb term written by*

$$V(x_2 - x_1) = \frac{a_c}{2}(x_2 - x_1) + V_C(x_2 - x_1). \quad (3.104)$$

Proof. By the diagonalization of Eq. (3.99) using a diagonal matrix $R_{kk'}$ associated with the eigenvalues $\eta_{(\mu)k}$ in the general case

$$S_Q = \sum_a \sum_k \eta_{(\mu)k} (A_{(Q)\mu k}^a)^2, \quad (3.105)$$

with

$$A_{(Q)\mu k}^a := R_{kk'} A_{(Q)\mu k'}^a, \quad (3.106)$$

it follows that

$$\begin{aligned} W_Q &= \text{Tr} \left\{ W_C I^{(3)} \frac{1}{Z_N} \int D[A_{(Q)\mu k}^a] \exp(-i \sum_a \sum_k B_{\mu k}^a A_{(Q)\mu k}^a) \right. \\ &\quad \left. \times \exp(-\sum_a \sum_k \eta_{(\mu)k} (A_{(Q)\mu k}^a)^2) \right\}, \end{aligned} \quad (3.107)$$

where

$$B_{\mu k}^a := g \sum_{k'} \beta_{\mu k'} R_{kk'}^{-1} T^a. \quad (3.108)$$

The integrals have the form of Gaussian integrals, and the odd term vanishes independently of the group, leading to

$$W_Q = \text{Tr} \left\{ W_C I^{(3)} \frac{1}{Z_N} \Pi_a \Pi_k \int dA'_{(Q)\mu k} \right. \\ \left. \times \cos(B'_{\mu k} A'_{(Q)\mu k}) \exp[-\eta_{(\mu)k} (A'_{(Q)\mu k})^2] \right\}. \quad (3.109)$$

By denoting $B'_{\mu k}$ as $B'_{\mu k} = \tilde{B}'_{\mu k} T^a$, and with the aid of

$$Z_N = \Pi_a \Pi_k \left(\frac{\pi}{\eta_{(\mu)k}} \right)^{1/2}, \quad (3.110)$$

the quantum Wilson loop results in

$$W_Q = \text{Tr} \left[W_C I^{(3)} \exp \left(- \sum_a \sum_k \frac{(\tilde{B}'_{\mu k})^2}{4\eta_{(\mu)k}} T^a T^a \right) \right]. \quad (3.111)$$

The sum of the matrices $T^a T^a$ is proportional to the unit matrix, and is independent of the group, showing that the quantum potential is a Coulomb potential. This is confirmed by the fact that the quantum potential of the Abelian field is a Coulomb potential. The analytical confining potential derived above is a sum of a classical linear term and quantum Coulomb term, as expressed by Eq. (3.104) \square

The above procedures also lead to the following theorem.

Theorem 3.28. *In pure Yang-Mills fields, the classical localized field as a vacuum produces the linear confinement potential between a charge and an anticharge, which are created by the pure Yang-Mills field. A non-zero binding energy between the paired charges generates a finite positive mass of the pure Yang-Mills field, namely, the mass gap, as shown in Eq. (3.84). Furthermore, a non-zero classical field gives rise to a local mass gap with non-zero positive value for the Yang-Mills field, as shown in Eq. (3.95). These masses give an answer to the mass gap problem of the Millennium problem.*

4 Conclusions

Despite usual mathematical papers contain no conclusion section, this section describes the conclusions. This paper presented a mathematical solution to the Yang-Mills existence and mass gap problem, which is one of the Millennium problems. The problem is widely known as a longstanding important problem. This paper has aimed at constructing solid mathematical foundations of field theory. The solution comprises two theorems derived via processes constructing associated sequential lemmas. For the first theorem, this paper formulated mathematically well-defined pure Yang-Mills fields, and a cutoff was introduced. Due to the asymptotic freedom, interactions of Yang-Mills fields vanish for the infinitely high-energy cutoff, meaning that Yang-Mills theory is trivial for this infinite energy. This paper then introduced an ultraviolet cutoff to avoid the triviality. In mathematics like for the Reimann space, it does not necessarily seem that the metric of general relativity should be excluded. Only in such the case for Yang-Mills fields, the cutoff scale is restricted to the Planck scale, preventing unphysical losses of high-energy intermediate states into black holes.

Fields are expanded in terms of scalar plane-waves, which are Poincaré/Lorentz covariant/invariant. The expansion coefficients have a symmetry represented by a compact and simple group. Below the cutoff energy, the momentum is continuous. Only as applications of mathematics for suppressing quadratic self-energies due to Higgs fields in physics, mathematics will provide a tool by dividing the four-dimensional spacetime continuum into arbitrarily shaped elements. Quantities, which correspond physically to vacuum expectation values for operators, have been analyzed within the regime below the cutoff energy. Corresponding mathematical formulations constructed from axioms have a basis in the traditional physical field theory except for high energies. The first theorem for this Poincaré/Lorentz covariant/invariant formalism without ultra-violet divergences presents an answer to the existence problem. Next, the pure Yang-Mills field considered is composed of classical stationary and quantum fluctuation fields. The classical field, which is a vacuum, has localized and unlocalized fields. From the Wilson loop for the classical localized field, there is a linear potential between a charge and an anticharge, and the binding energy yields a mass gap, by stabilizing the system compared to the Coulomb phase. Meanwhile, quantum fluctuations add a Coulomb potential to the linear potential. In the action, a local mass term appears due to the classical field. Due to these masses, the second theorem presents an answer to the mass gap problem. Consequently, the Yang-Mills existence and mass gap problem has been solved.

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Appendix A : Aim and outline of Appendix B and Appendix C

This Appendix A describes the background, aim, and outline of Appendix B and Appendix C. Appendix B and Appendix C are aimed at describing a formulation only for a possible case of mathematics applications to physics. In these Appendices, a cutoff at the high energy is denoted by Π_H , which means the cutoff introduced in Sec. 2 in the main procedure of this paper. Whereas, a possible cutoff at the energy lower than the above cutoff Π_H is denoted by I_L , which is introduced in these Appendices. The low-energy cutoff I_L suppresses the extreme large quadratic self-energy by Higgs fields when the energy of the high-energy cutoff Π_H is inescapably high. The low-energy cutoff I_L is achieved by dividing the 4-dimensional spacetime continuum into arbitrary-shaped spacetime elements. Although the high-energy cutoff Π_H makes physical quantities finite for self-interactions, this cutoff is insufficient. This is because when interactions include self-interactions to some extent contributed from Higgs particles around 10^{11} eV (10^2 GeV), the calculated results with the high-energy cutoff Π_H become extremely large values, which are unphysical.

In a lattice theory, fields are defined on a discrete lattice in the spacetime continuum, and wave lengths shorter than the lattice spacing are cut off. By contrast, it is widely known that in solid-state physics, the cutoff is achieved by dividing the space continuum into periodically aligned space elements [25, 26]. For the nearest-neighbor distance a_s between space elements, wave lengths shorter than a_s , which corresponds to the absolute momentum of π/a_s , are cut off in this case. Additionally, without influencing this cutoff, the

wave rarely absorbs or emits a discrete momentum $G_n = (2\pi/a_s)n$, with n being an integer variable, raising or lowering the wave energy. The calculation results for solids obtained using this band theory agree well with experimental results.

By analogy with solid-state physics, this paper deals with the low-energy cutoff I_L . The formalism is required to satisfy the Poincaré covariance with the rotational and translational covariance. This theory divides the four-dimensional (4D) spacetime continuum into arbitrary-shaped spacetime elements. Fields are defined on this spacetime divided into elements that are locally periodically aligned. The momentums with the individual absolute value larger than π/a_r are cut off, where a_r denotes the nearest-neighbor distance between the spacetime elements. Additionally, it is known that a function with the same periodicity as the space elements is expressed using the Fourier series as a function of momentum $G_n = (2\pi/a_r)n$, where n is an integer variable [26]. Here, this theory introduces in Eq. (C.5) a compensation field with the same periodicity as that of the space elements. The momentum of the field is $G_{n_\mu} = (2\pi/a_r)n_\mu$ with n_μ being integers for coordinate indices $\mu = 0, 1, 2, 3$. For the local periodicity, the momenta are given by Eq. (C.3). Without influencing the low-energy cutoff I_L , the field exchanges momentum $G_{n_\mu} = (2\pi/a_r)n_\mu$ via interactions with the compensation field, and rarely absorbs/emits energy within the energy range of the field up to the energy of the high-energy cutoff I_H . Because of the long-range disorder, the wave is scattered by the exchange of momentum in Eq. (C.3). As mentioned in Sec. 2, using the plane waves in Eq. (2.10) with the cutoff, the fields with the cutoff are expressed in the form of the Fourier expansion, as shown in Eq. (2.11). It is noted that as mentioned in Sec. 2, the underlining for quantities with the cutoff is not indicated, except in cases where it is explicitly required to denote the cutoff. Physics describes nature, while mathematics is independent of nature and is based on axioms. In other words, mathematics is not necessary to describe nature. Nevertheless, physically the low-energy cutoff I_L based on the relationships between some physical quantities was introduced in the literatures [19, 20]. The energy of the low-energy cutoff I_L is about 3×10^{15} eV (3 PeV), which is much lower than the energy of the high-energy cutoff I_H . In fact, such as a break (abnormal behavior, the so-called knee) in the energy dependence of natural cosmic-rays has been experimentally observed around 3 PeV, as reported in the literatures [37, 38] and other ones [39, 40].

Appendix B describes the cutoff by analogy with solid-state physics. The 4D spacetime continuum is divided into arbitrarily shaped spacetime elements that are aligned locally periodically. Wave functions are defined on this spacetime with divided spacetime elements, and the wave length of the wave function is restricted to that longer than the nearest-neighbor distance between the spacetime elements. In this paper, this cutoff at the low energy is denoted by I_L , and is distinguished from the high-energy cutoff I_H .

In Appendices, Poincaré covariant plane waves are also used for a Fourier expansion as a function of four-momentum within the specified cutoff energy, because the cutoff in this paper is represented in 4D momentum space.

Appendix C presents the low-energy cutoff energy, which is (used in physics as the same meaning as) the reciprocal of the cutoff length that is the nearest-neighbor distance between spacetime elements. Additionally, the divided spacetime elements are aligned locally periodically. It is known that a periodic function in the spacetime with this periodicity is expressed by a Fourier series as a function of the momentum that is the product of the cutoff momentum (vector) and integers, as shown in Eq. (C.3). A compensation field with the this momentum is then introduced in Eq. (C.5) without influencing the low-energy cutoff I_L . A field particle is only rarely excited by the above effective compensation field beyond the energy of the low-energy cutoff

I_L through interactions with the compensation field. Furthermore, the field particle is not excited above the energy of the high-energy cutoff Π_H . Yang-Mills fields are expressed in the form of Fourier expansion in terms of the plane waves as a function of the momentum within the energy of the low-energy cutoff I_L , as described before in Definition 2.5.

Appendix B: Spacetime continuum divided into spacetime elements

In this Appendix B, the author shows a possible formulation only for the case, in which an application of mathematics to physics is considered. In Sec. 2, the cutoff, denoted by Π_H hereafter, of the main procedure at the high energy was formulated for quantum field theory (QFT) without ultraviolet divergences. Besides the above high-energy cutoff Π_H , the cutoff denoted here by I_L at a low energy is introduced to suppress quadratic self-energies associated with Higgs fields. Fields with state vectors are defined mainly on the four-dimensional (4D) Euclidean spacetime continuum \mathbb{R}_E^4 . As aforementioned in Appendix A, the present Appendix B describes the low-energy cutoff I_L obtained by dividing the 4D spacetime continuum into arbitrary-shaped spacetime elements aligned locally periodically. This system is based on solid state physics, in which the wave lengths are restricted to those longer than the nearest-neighbor distance between spacetime elements, as described in Appendix C.

The spacetime continua \mathbb{R}_E^4 and \mathbb{R}_M^4 are divided into spacetime elements, in which hypersurfaces normal to the time-axis at each point are space-like. The spacetime elements are aligned locally periodically without any long-range order. These are described by the following Definitions Appendix B.1-Appendix B.3.

Definition Appendix B.1. Let \mathbb{R}_P^4 be the four-dimensional Euclidean parameter spacetime continuum. Then its coordinates are denoted by $x_{P\mu} := (x_{P0}, x_{P1}, x_{P2}, x_{P3}) = (t_P, x_P, y_P, z_P)$. For $k, l, m, n \in \mathbb{N}^+$ with \mathbb{N}^+ being positive natural numbers, let p be indices denoted by $p = (k, l, m, n)$. The parameter spacetime \mathbb{R}_P^4 is divided into hypercubes, then a lattice/grid point in the parameter spacetime is defined by $x_{Pp} = x_{P(k,l,m,n)} := (t_{P(k)}, x_{P(l)}, y_{P(m)}, z_{P(n)})$.

Definition Appendix B.2. Let \mathbb{R}_M^4 and \mathbb{R}_P^4 be the 4D Minkowski spacetime and Euclidean parameter spacetime, respectively. For four-vectors $x_P \in \mathbb{R}_P^4$ and $x \in \mathbb{R}_M^4$, the following map is introduced

$$f_P : x_P \in \mathbb{R}_P^4 \mapsto x \in \mathbb{R}_M^4, \quad (\text{B.1})$$

where each component of the above map is written by

$$x_\mu = f_{P\mu}(x_P) = f_{P\mu}(t_P, \mathbf{x}_P). \quad (\text{B.2})$$

The inverse map is defined by

$$f_R = f_P^{-1} : x \in \mathbb{R}_M^4 \mapsto x_P \in \mathbb{R}_P^4. \quad (\text{B.3})$$

Definition Appendix B.3. For $(t_P, \mathbf{x}_P) \in \mathbb{R}_P^4$ and $(t, \mathbf{x}) \in \mathbb{R}_M^4$, a three-dimensional hypersurface σ_{P_t} normal to the t_P -axis in \mathbb{R}_P^4 is introduced, and then σ_t in \mathbb{R}_M^4 is defined as a three-dimensional hypersurface mapped from σ_{P_t} under the function in Eqs. (B.1)–(B.3). For $x_\mu, x'_\mu \in \mathbb{R}_M^4$ on the space-like hypersurface σ_t at the point x_μ , the following relation [5, 9, 10, 15] is satisfied

$$(t' - t)^2 \leq (x'_i - x_i)^2. \quad (\text{B.4})$$

Appendix C: Cutoff at the energy lower than the ultraviolet cutoff energy solely for possible applications of mathematics to physics

Here, let us define two Poincaré invariant/covariant cutoffs. Only for possible applications of mathematics to physics, the low-energy cutoff I_L at the energy lower than that of the high-energy cutoff Π_H is effective at removing the large quadratic self-energies associated with the Higgs field. By analogy with solid state physics as described in Appendix A, the low-energy cutoff I_L is achieved owing to the division of the spacetime continuum into spacetime elements. In this case, the wave length is longer than the size of the spacetime element. As a result, continuous momentums of the usual Yang-Mills field are restricted to the regime within the energy of the low-energy cutoff I_L . On the other hand, it is stated in the literature [26] that when a function on the Euclidean spacetime has the same local periodicity as that of the local alignment of the spacetime elements, the function has Fourier components of the discrete momentum; and this discrete momentum is the reciprocal cutoff four-momentum multiplied by integers. Here, the present theory introduces an effective compensation vector field in the form of this periodic function with discrete momentums. By interactions with the effective compensation field, quantum Yang-Mills gluon particles are only rarely excited beyond the energy of the low-energy cutoff I_L . Furthermore, above the energy of the low-energy cutoff I_L , the author uses the cutoff Π_H introduced in Sec. 2 at the high energy.

Definition Appendix C.1. Let $a_r \in \mathbb{R}$ be the separation between each center of the nearest-neighbor spacetime elements, whose alignment is locally periodic without any long-range order. Four-momentums are limited within in the following first momentum zone by the cutoff

$$|\underline{k}| := |\underline{k}_\mu| = |(\underline{k}_\mu \underline{k}_\mu)^{1/2}| \leq \pi/a_r, \quad (\text{C.1})$$

and plane waves with the cutoff are expressed as

$$\underline{\chi}_{\underline{k}} := \exp(\underline{k}_\mu x_\mu). \quad (\text{C.2})$$

For the quantities whose momentums are limited within the cutoff energy, an underline under the symbol(s) such as $\underline{\chi}$ and \underline{k} is added, although this underline is not always denoted explicitly.

The underline is added explicitly only if necessary. The momentums beyond the energy of the high-energy cutoff Π_H are also cut off.

Definition Appendix C.2. Let a_r be the separation between spacetime elements in the above Definition Appendix C.1. Let $x_p = x_{p\mu} = (x_{p0}, x_{p1}, x_{p2}, x_{p3})$ be a center of the space-time element. For integers n_0, n_1, n_2, n_3 , quantities $G_{\bar{n}} := G_{\mu\bar{n}} := G_{\mu\bar{n}p}$ at x_p are defined by

$$\begin{aligned} G_{0\bar{n}p} &:= n_0(2\pi/a_r)e_{0p}, & G_{1\bar{n}p} &:= n_1(2\pi/a_r)e_{1p}, \\ G_{2\bar{n}p} &:= n_2(2\pi/a_r)e_{2p}, & G_{3\bar{n}p} &:= n_3(2\pi/a_r)e_{3p}. \end{aligned} \quad (\text{C.3})$$

Here, $e_{\mu p}$ at x_p are unit vectors, which are mutually orthogonal, and are oriented in arbitrary directions.

Lemma Appendix C.3. On the spacetime continuum \mathbb{R}_E^4 , which is divided into spacetime elements aligned with a local periodicity without long-range order, let us consider a function $u(x)$, which has the same periodicity as the aligned spacetime elements. Then $u(x)$ is expanded in the form of the following Fourier series

in terms of $G_{\mu\bar{n}}$ in Definition Appendix C.2

$$u(x) = \sum_{G_{\bar{n}}} u_{G_{\bar{n}}} \exp(iG_{\mu\bar{n}}x_{\mu}) \quad \text{with } G_{\bar{n}} := G_{\mu\bar{n}}. \quad (\text{C.4})$$

The proof of the above lemma is given in Chap. 1 of the literature [26].

Here, the following effective compensation vector field is newly introduced.

Definition Appendix C.4. Using the four-momentums $G_{\nu\bar{n}}$ in Definition Appendix C.2, let us introduce the following compensation vector field, which has the same local periodicity as the periodically aligned spacetime elements that are obtained by dividing the spacetime continuum \mathbb{R}_E^4 ,

$$A_{(\text{EC})\mu}(x) := \sum_{G_{\bar{n}}} A_{(\text{EC})\mu G_{\bar{n}}} \exp(iG_{\nu\bar{n}}x_{\nu}), \quad (\text{C.5})$$

where $A_{(\text{EC})\mu G_{\bar{n}}}$ are expansion coefficients.

Definition Appendix C.5. Only via interactions with the effective compensation field of discrete momentum in Definition Appendix C.4, a particle described by a usual plane wave is rarely excited by an additional discrete momentum. The plane wave then has the following form

$$\begin{aligned} \chi_{kG_{\bar{n}}}(x) &:= \sum_{G_{\bar{n}}} \exp[i(\underline{k}_{\mu} + G_{\mu\bar{n}})x_{\mu}] \\ &= \sum_{G_{\bar{n}}} \exp(iP_{\mu}x_{\mu}). \end{aligned} \quad (\text{C.6})$$

In the above equation the quantity \underline{k}_{μ} is the four-momentum in the first momentum zone given by Eq. (C.1) using a_r in Definition Appendix C.1. Additionally, the momentum P_{μ} of the quantum particle becomes $P_{\mu} = \underline{k}_{\mu} + G_{\mu\bar{n}p}$.

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