A mathematical solution to Yang-Mills existence and mass gap problem, one of the Millennium problems

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In this article, the Poincaré/Lorentz covariant/invariant formalism without ultraviolet divergences gives an answer to the Yang-Mills existence problem of the longstanding Millennium problem (MP), since axiomatic approaches use traditional field properties. The first cutoff, corresponding physically to the cutoff of quadratic self-energies caused by Higgs fields, divides the four-dimensional spacetime continuum into arbitrarily shaped elements. Fields are expanded using scalar plane-waves with continuous four-momentum of quantum particles. The local alignment of spacetime elements is periodic without long-range order. Then, an effective field with discrete momentums is introduced. Particles are rarely excited in interactions with this effective field. The higher energy of the second cutoff corresponds physically to the Planck energy, preventing unphysical losses of high-energy intermediate states into black holes. Quantities, which correspond physically to vacuum expectation values of operators, utilize the first cutoff. Next, pure Yang-Mills fields composed of variational stationary classical fields and quantum fluctuations are considered. The Wilson loop for the classical field yields a linear potential between a charge and an anticharge, and the stabilization energy provides a mass gap. The action has a local mass from the classical field. These masses present an answer to the mass gap problem of the MP.

1 Introduction and Summary

This paper presents a mathematical solution to the Yang-Mills (YM) existence and mass gap problem, which is one of the Millennium problems [1]. The problem is widely known as a longstanding important problem to be solved. The aim of science is to clarify unclear problems. In this paper, Sec. 2 presents a mathematical solution to the Yang-Mills (YM) existence problem, deriving Theorem 2.20. Section 3 gives a solution to the mass gap problem, leading to Theorem 3.28. Subsections 2.1 and 3.1 describe the aim and outline of Secs. 2 and 3, respectively. This study aims at constructing solid foundations of quantum field theory (QFT) for Yang-Mills fields [2]. Mathematics is a science which constructs axioms and which derives theorems from the axioms. Although mathematics is independent of physics, mathematics and physics have mutually influenced each other. In physical field theory, an expectation value for a product of operators [2–6]

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sometimes diverges due to high-energy contributions. This is known as the ultraviolet divergence problem, [7–15] which this paper treats. Mathematically, this expectation value with ultraviolet divergences may be in some sense expressed by a tempered distribution. This distribution is included in Schwartz distributions, and its Fourier transforms are available. In physical field theory, there are two kinds of problems caused by high-energy interactions. One is the ultraviolet divergence, whose dominant contribution arises from quadratic self-energies associated with Higgs fields [16] in the relatively low-energy regime. The other occurs in interacting intermediate states for the relatively high-energy regime beyond the Planck energy, which is the fundamental unit energy. In the latter case, probabilities associated with quantum particles are unphysically lost into black holes. On the other hand, non-Abelian Yang-Mills fields should physically gain a mass, which leads to a transition from long-range interactions to short-range interactions. A set of these mathematical requirements is the Yang-Mills existence and mass gap problem. A physical formulation of the themes including a solution to the above problems has been presented in previous papers [17–21]. Other approaches in different motivations are seen in the literatures [22–24].

On the basis of this physical formulation, this paper aims at giving a mathematical solution to the Yang-Mills existence and mass gap problem. Tomonaga (S. Tomonaga) stated that when the problem is difficult, one way is to learn theoretical schemes from nature. For this reason, two theories with associated lemmas in this paper are constructed mathematically, referring to physical meanings. First, in Sec. 2, a mathematically well-defined field theory is formulated, presenting Theorem 2.20. In the formalism, two cutoffs are introduced, and for the first cutoff at a relatively low energy, the four-dimensional (4D) spacetime continuum is divided into arbitrarily shaped spacetime elements. The alignment of these spacetime elements is locally periodic without long-range order, and element surfaces normal to the local time are space like. Fields are expanded in terms of scalar expansion functions, which are plane waves as a function of an inner product between a four-vector in the 4D spacetime and a four-momentum. The fields are Poincaré covariant under rotational and translational transformations in the spacetime continuum. The expansion coefficients of the Yang-Mills gauge field have a symmetry represented by such as a compact and simple group [2]. The four-momentum is continuous for energies lower than the first cutoff energy. Due to the local periodic alignment of spacetime elements [25, 26], an effective field with discrete momentums is introduced. Quantum gluon particles of pure Yang-Mills fields are rarely excited only via interactions with this effective field beyond the first cutoff energy. To avoid missing of probabilities with particles beyond the energy corresponding physically to the Planck energy, the second cutoff is introduced at this energy. Meanwhile, quantities, corresponding physically to vacuum expectation values for a product of operators, are formulated using the first cutoff. The Lagrangian in the action is defined in a traditional way for the cutoff scheme, and the fields are quantized using the path integral. The formalism that presents Theorem 2.20 constructs Poincaré/Lorentz covariant/invariant Yang-Mills fields without ultraviolet divergences, and is consistent with the corresponding properties of the axiomatic formalism [5] by Streater & Wightman (SW) or equivalently that by Osterwalder & Schrader (OS). It is emphasized that except for high-energy contributions, which give rise to the ultraviolet divergence problem, the formalism of physical field theory is already mathematically rigorous. The field theory defined in theoretical physics is therefore not repeated mathematically here.

Next, in Sec. 3, a solution is presented to the mass gap problem for pure Yang-Mills fields, stated by Theorem 3.28. The quantization scheme employed in this paper is the Feynman path integral. It is regarded that contributions to the path integral are a variational stationary classical field and quantum fields. The former classical field leads to a differential equation in analytic dynamics, and the latter quantum fields fluctuate around the classical field. The pure Yang-Mills field then comprises classical and quantum fields. An example of compact simple groups considered here is represented by such as SU(N) of degree N, which does not involve any loss of generality. The classical localized field with SU(2) symmetry is embedded in the classical SU(N) field with all zero elements. The classical field considered in this study is the sum of localized and unlocalized fields. For a specific configuration of the classical fields, the classical Yang-Mills differential equation is then reduced to a linear equation. This linear equation differs from that for the $U(1)$ gauge field, because each component of the Yang-Mills field is multiplied by a generator (matrix) of the group. The charge and anticharge densities associated with the localized and unlocalized fields, respectively, cancel each other. The classical localized field gives rise to a linear potential between a charge and an anticharge. Quantum fluctuations around the vacuum of the classical zero-field provide a Coulomb potential between paired charges. By contrast, when the classical field contains the localized field, a linear potential appears, and quantum fluctuations add the Coulomb potential to the linear potential. The latter confinement phase, which yields the binding energy between the paired charges under the linear potential, is stable compared to the former Coulomb phase. Moreover, the non-interacting term in the Lagrangian of the action gains a positive local mass originating from the non-vanishing classical field. Thus, in Theorem 3.28, not only the above binding energy between the paired charges but also the local mass contributes to the mass gap of the pure Yang-Mills field. Hence, this paper presents a solution to the Yang-Mills existence and mass gap problem.

This paper is organized as follows. Subsections 2.1 and 3.1 outline Secs. 2 and 3, respectively, for easier understanding. Then, the next Sec. 2 presents a well-defined formulation of Yang-Mills fields without ultraviolet divergences, leading to Theorem 2.20. Subsequently, Sec. 3 gives a solution to the mass gap problem of the pure Yang-Mills field, resulting in Theorem 3.28. Although usual mathematical papers need no conclusion section, Sec. 4 in this paper summaries conclusions.

2 A solution to the Yang-Mills existence problem: Formulation of quantum field theory without ultraviolet divergences

2.1 Aim and outline of Sec. 2

This Subsec. 2.1 describes an outline of this Sec. 2. Quantum field theory in physics is rigorous, except for the problem of ultraviolet divergences. This section then presents a Poincaré covariant field theory without ultraviolet divergences, leading to a solution to the Yang-Mills (YM) existence problem. The remainder of this subsection describes the background, aim, and outline of this section.

Quantum field theory, which is quantized equivalently by such as the path integral or canonical quantization, has too many high-energy configurations (values), resulting in the ultraviolet divergence problem. Therefore, the ultraviolet divergences must be removed. Using plane waves in Eq. (2.18) with coordinates x_μ and energy-momentum (four-momentum) p_μ on the four-dimensional spacetime continuum, a field without the cutoff of high energies (including momentums) is expressed in a Fourier expansion form of Eq. (2.19), where T^a denote generators of the group such as SU(N) with N being a general and meaningful natural number. When a quantum particle with unlimited energies emitted from a field is reabsorbed by the field in the self-interacting intermediated state, ultraviolet divergences occur in some physical quantities. However, when the energy of the quantum particle exceeds the Planck energy, which is the high-energy fundamental unit of about 10^{28} eV (10^{19} GeV), the particle becomes a black hole. This particle loss is unphysical in the intermediate state. Therefore, it is defined in this paper that the upper bound of the field energy is the Planck energy to prevent the unphysical particle loss. Here, the cutoff of the energy here is named high-energy (second) cutoff. Although the high-energy (second) cutoff makes physical quantities finite for self-interactions, this cutoff is insufficient. This is because when interactions include self-interactions to some extent contributed from Higgs particles around 10^{11} eV (10^2 GeV), the calculated results with the high-energy (second) cutoff become extremely large values, which are unphysical.

In a lattice theory, fields are defined on a discrete lattice in the spacetime continuum, and wave lengths shorter than the lattice spacing are cut off. By contrast, it is widely known that in solid-state physics, the cutoff is achieved by dividing the space continuum into periodically aligned space elements [25, 26]. For the nearest-neighbor distance a_{s} between space elements, wave lengths shorter than a_{s}, which corresponds to the absolute momentum of π/a_{s}, are cut off in this case. Additionally, without influencing this cutoff, the wave rarely absorbs or emits a discrete momentum $G_n = (2\pi/a_{\mathrm{s}})n$, with n being an integer variable, raising or lowering the wave energy. The calculation results for solids obtained using this band theory agree well with experimental results.

By analogy with solid-state physics, this paper deals with the cutoff named low-energy (first) cutoff. For field theory, the Poincaré covariance with the rotational and translational covariance is required. This theory divides the four-dimensional (4D) spacetime continuum into arbitrary-shaped spacetime elements. Fields are defined on this spacetime divided into elements that are locally periodically aligned. The momentums with the individual absolute value larger than π/a_{r} are cut off, where a_{r} denotes the nearest-neighbor distance between the spacetime elements. Additionally, it is known that a function with the same periodicity as the space elements is expressed using the Fourier series as a function of momentum $G_n = (2\pi/a_{\mathrm{r}})n$, where n is an integer variable [26]. Here, this theory introduces in Eq. (2.13) a compensation field with the same periodicity as that of the space elements. The momentum of the field is $G_{n_\mu} = (2\pi/a_{\mathrm{r}})n_\mu$ with n_μ being integers for coordinate indices $\mu = 0, 1, 2, 3$. For the local periodicity, the momenta are given by Eq. (2.11). Without influencing the low-energy (first) cutoff, the field exchanges momentum $G_{n_\mu} = (2\pi/a_{\mathrm{r}})n_\mu$ via interactions with the compensation field, and rarely absorbs/emits energy within the energy range of the field up to the Planck energy, which is the higher (second) cutoff energy. Because of the long-range disorder, the wave is scattered by the exchange of momentum in Eq. (2.11). Thus, using the plane waves in Eq. (2.20) with the cutoff, the fields with the cutoff are expressed in the form of the Fourier expansion, as shown in Eq. (2.21). It is noted that in this paper, the underlining for quantities with the cutoff is omitted, except in cases where it is explicitly required to denote the cutoff. Physics describes nature, while mathematics is independent of nature and is based on axioms. In other words, mathematics is not necessary to describe nature. Nevertheless, physically the low-energy (first) cutoff based on the relationships between some physical quantities was introduced in the literatures [20, 21]. The energy of the first cutoff is about 3×10^{15} eV (3 PeV), which is much lower than the Planck energy. In fact, such as a break (abnormal behavior, the so-called knee) in the energy dependence of natural cosmic-rays has been experimentally observed around 3 PeV, as reported in the literatures [27, 28] and other ones [29, 30].

To provide an overview of the cutoff in quantum field theory calculations mentioned below, quantum theory used in this paper is briefly summarized here, with reference to [2–6, 9–15]. In principle, field theory is formulated using a differential equation reproduced from a stationary state with a vanishing variation of the action functional (Lagrangian) in analytical dynamics. One quantization theory is Feynman path integral, in which integrals are performed over all field configurations (field function values) associated with the weight of the action. Canonical quantization of the operator form is another theory that was developed from the Hamilton formalism in analytic dynamics. As reported in the literatures [13, 14], a state vector $|\Psi>$, which represents a field system, changes to $|\Psi'>$ by the multiplication of matrix U_{I} in interactions, as shown in Eq. (2.28) in Subsec. 2.5. The probability (transition) amplitude in Eq. (2.29) projected onto the vector $<\Psi''|$ is derived from $<\Psi''|\Psi'>=<\Psi''|U_{\mathrm{I}}|\Psi>$. When a coupling constant in interactions is small like that of the Yang-Mills field with the well-known asymptotic freedom [2] mentioned later, an expansion of U_{I} is performed using the Feynman-Dyson diagram method. It is known that in path integral, derivatives of a generating functional yield terms in a power series of U_{I} with respect to a coupling constant. [2, 3]. Using the Fourier expansion of fields as a function of four-momentum, field theory calculations are performed in momentum space. In the Feynman diagram, as shown later in Figure 1 in Subsec. 2.5, a straight or curved line connecting two intersections of lines physically corresponds to the vacuum expectation value for a time-ordered product of field operators (in the interaction representation). This expectation value, so-called such as the invariant function, propagator, Green function, correlation function, or Wightman function, is typically a tempered distribution. From the Wightman function in the Minkowski space, Schwinger function in the Euclidean space is derived through a transformation on a complex plane. In integrals in 4D momentum space according to the Feynman diagram, high-energy contributions to the diagrams, such as so-called loop and vertex, yield ultraviolet divergences. Proper diagrams, which cannot be essentially divided, are separated, including the case of a higher-order series. The ultraviolet divergences are removed by cutting off high-energy contributions. Notably, the characteristics of this paper employ the Lorentz-invariant cutoff radius, which is rotationally (Lorentz) invariant in the Euclidean space. The divergent integrals described by the proper diagrams are put into a set of limited number of renormalization constants (including subtraction cases), and fields are renormalized. Due to this renormalization, the coupling constant depends on the cutoff energy. Yang-Mills fields then reveal the asymptotic freedom, which is the property that the coupling constant decreases as the cutoff energy increases.

The Poincaré covariant cutoff in this paper is used in the following procedure. The Feynman-Dyson diagram method developed in the original studies [12–14] are described in detail in many literatures [5, 15]. Figure 1 depicted later in Subsec. 2.5 shows a two-point graph for a self-interaction in the Feynman diagram, representing a two-point function below. In this graph, the process $x \to y$ is finally taken. Although this graph exactly corresponds to Eq. (2.39), the conceptual meaning is roughly symbolized in Eq. (2.30). The general term displayed by the Feynman diagram is given by Eq. (2.37), which can be rewritten as a simple product of the two-point functions, as indicated in Eq. (2.38). In a sample case, Eq. (2.38) becomes simple Eq. (2.39) followed by Eq. (2.40). In the diagram, a quantum particle with a momentum p (namely p_μ) annihilates at intersection y and a particle with momentum k is created. This particle propagates along a curved line and annihilates at intersection x. Here, a particle with the momentum p is created. In this case, the process $x \to y$ is finally taken to yield a quartic interaction, in which the field reabsorbs the emitted particle at the same intersection. The field particle without the cutoff propagates freely between the two intersections in the intermediate state during the interaction and takes any momentum k. Conversely, the momentum with the cutoff is restricted within the cutoff energy.

Here, the remainder of this Subsec. 2.1 sequentially overviews the theoretical procedures in the current Sec.

2, summarizing each subsection and referring to some equations.

Section 2.2 describes the cutoff by analogy with solid-state physics. The 4D spacetime continuum is divided into arbitrarily shaped spacetime elements that are aligned locally periodically. Wave functions are defined on this spacetime with divided spacetime elements, and the wave length of the wave function is restricted to that longer than the nearest-neighbor distance between the spacetime elements. In this paper, this cutoff is named low-energy (first) cutoff, and is distinguished from high-energy (second) cutoff at the Planck energy.

Subsection 2.3 describes a Poincaré covariant plane wave in Eq. (2.8), which is used for a Fourier expansion as a function of four-momentum, since the cutoff in this paper is also represented in 4D momentum space.

Subsection 2.4 presents the low-energy (first) cutoff energy, which is (used in physics as the same meaning as) the reciprocal of the cutoff length that is the nearest-neighbor distance between spacetime elements. Additionally, the divided spacetime elements are aligned locally periodically. It is known that a periodic function in the spacetime with this periodicity is expressed by a Fourier series as a function of the momentum that is the product of the cutoff momentum (vector) and integers, as shown in Eq. (2.11). A compensation field with the this momentum is then introduced in Eq. (2.13) without influencing the low (first) energy cutoff. A field particle is only rarely excited by the above effective compensation field beyond the low (first) cutoff energy through interactions with the compensation field. Furthermore, the field particle is not excited above the high (second) cutoff energy at the Planck energy to avoid the particle loss into a black hole. Yang-Mills fields are expressed in the form of Fourier expansion in terms of the plane waves as a function of the momentum within the low (first) cutoff energy, as described in Definition 2.13.

In Subsec. 2.5, the beginning up to around Eq. (2.40) is devoted to summarizing the Feynman-Dyson diagram method, which gives an expansion in powers of a coupling constant, for field theory calculations. A (vacuum) expectation value of a product of operators is given in the Minkowski space in Eq. (2.40). Using the residue theorem for complex functions, the expectation value is rewritten to the form in the Euclidean momentum space, as shown in Eq. (2.41). The field without the cutoff yields divergences of self-interaction quantities contained in the transition amplitude as shown in Eq. (2.44), while the field with the rotationally invariant cutoff gives a finite value without ultraviolet divergences, as indicated in Eq. (2.45).

Subsection 2.6 refers to the relationship between (vacuum) expectation values in the Minkowski spacetime and tempered distributions, showing the energy cutoff.

The lemmas and the theoretical schemes in this Sec. 2 finally lead to Theorem 2.20, which is a solution to the Yang-Mills existence problem.

2.2 Spacetime continuum divided into spacetime elements

As mentioned in the previous Sec. 1, mathematical theories, which are independent of physics, are constructed in this paper, referring to physical formulations with their meanings. In this section, quantum field theory (QFT) without ultraviolet divergences is formulated by introducing two cutoffs. Fields with state vectors are defined mainly on the four-dimensional (4D) Euclidean spacetime continuum $\mathbb{R}^4_{\mathrm{E}}$. As aforementioned in Subsec. 2.1, the present subsection describes the low-energy (first) cutoff obtained by dividing the 4D spacetime continuum into arbitrary-shaped spacetime elements aligned locally periodically. This system is based on solid state physics, in which the wave lengths are restricted to those longer than the nearest-neighbor distance between spacetime elements, as described in Subsec. 2.4.

Definition 2.1. For a set of real numbers $\mathbb{R}$, let $t \in \mathbb{R}$ and $x, y, z \in \mathbb{R}$ be time and space coordinates, respectively. Physically, $c = 1$ and $\hbar := h/\pi = 1$ in natural units, where c and h correspond to the speed of light and the Planck constant, respectively. Let $\mathbb{R}^4 = \{x^\mu\}$ with $\mu = 0,1,2,3$ be 4D spacetime continuum, where x^μ are contravariant components of a four-vector, denoted by $x^0 := t = ct$, $x^1 := x$, $x^2 := y$, $x^3 := z$. It is noted that x used as a coordinate is also used as a four-vector denoted by $x = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}) = (x^0, \mathbf{r}) = (t, x, y, z)$. Using a repeated Roman index i with $i = 1,2,3$, the summation of spatial components is expressed by $(x^i)^2 := x_i x^i = (x^1)^2 + (x^2)^3 + (x^3)^2$. Hereafter, the covariant and contravariant components are not distinguished, because there is no need to distinguish between them in the Yang-Mills theory. Let $\mathbb{R}^4_{\mathrm{M}}$ be the four-dimensional Minkowski spacetime continuum. For $x_\mu, y_\mu \in \mathbb{R}^4_{\mathrm{M}}$ with the repeated Greek index μ, which is a non-negative integer that runs from 0 to 3, the scalar inner product is defined by $x \cdot y := x_\mu y_\mu = -x_0 y_0 + x_i y_i$ instead of $x \cdot y = x_0 y_0 - x_i y_i$. Via Wick rotation, this product is transformed to that in the Euclidean spacetime continuum denoted by $\mathbb{R}^4_{\mathrm{E}} = \mathbb{R}^4$. Then, for $x_\mu, y_\mu \in \mathbb{R}^4_{\mathrm{E}}$, the scalar inner product and norm are defined by $x \cdot y := x_\mu y_\mu = x_0 y_0 + x_i y_i$ and $|x \cdot x| := ||x \cdot x|| := (x \cdot x)^{1/2}$, respectively.

The spacetime continua $\mathbb{R}^4_{\mathrm{E}}$ and $\mathbb{R}^4_{\mathrm{M}}$ are divided into spacetime elements, in which hypersurfaces normal to the time-axis at each point are space-like. The spacetime elements are aligned locally periodically without any long-range order. These are described by the following Definitions 2.2-2.4.

Definition 2.2. Let $\mathbb{R}^4_{\mathrm{P}}$ be the four-dimensional Euclidean parameter spacetime continuum. Then its coordinates are denoted by $x_{\mathrm{P}\mu} := (x_{\mathrm{P}0}, x_{\mathrm{P}1}, x_{\mathrm{P}2}, x_{\mathrm{P}3}) = (t_{\mathrm{P}}, x_{\mathrm{P}}, y_{\mathrm{P}}, z_{\mathrm{P}})$. For $k, l, m, n \in \mathbb{N}^+$ with $\mathbb{N}^+$ being positive natural numbers, let p be indices denoted by $p = (k, l, m, n)$. The parameter spacetime $\mathbb{R}^4_{\mathrm{P}}$ is divided into hypercubes, then a lattice/grid point in the parameter spacetime is defined by $x_{\mathrm{P}p} = x_{\mathrm{P}(k,l,m,n)} := (t_{\mathrm{P}(k)}, x_{\mathrm{P}(l)}, y_{\mathrm{P}(m)}, z_{\mathrm{P}(n)})$.

Definition 2.3. Let $\mathbb{R}^4_{\mathrm{M}}$ and $\mathbb{R}^4_{\mathrm{P}}$ be the 4D Minkowski spacetime and Euclidean parameter spacetime, respectively. For four-vectors $x_{\mathrm{P}} \in \mathbb{R}^4_{\mathrm{P}}$ and $x \in \mathbb{R}^4_{\mathrm{M}}$, the following map is introduced

$$f_{\mathrm{P}} : x_{\mathrm{P}} \in \mathbb{R}^4_{\mathrm{P}} \longmapsto x \in \mathbb{R}^4_{\mathrm{M}}, \tag{2.1}$$

where each component of the above map is written by

$$x_\mu = f_{P\mu}(x_{\mathrm{P}}) = f_{\mathrm{P}\mu}(t_{\mathrm{P}}, \mathbf{x}_{\mathrm{P}}). \tag{2.2}$$

The inverse map is defined by

$$f_{\mathrm{R}} = f_{\mathrm{P}}^{-1} : x \in \mathbb{R}^4_{\mathrm{M}} \longmapsto x_{\mathrm{P}} \in \mathbb{R}^4_{\mathrm{P}}. \tag{2.3}$$

Definition 2.4. For $(t_{\mathrm{P}}, \mathbf{x}_{\mathrm{P}}) \in \mathbb{R}^4_{\mathrm{P}}$ and $(t, \mathbf{x}) \in \mathbb{R}^4_{\mathrm{M}}$, a three-dimensional hypersurface $\sigma_{\mathrm{P}t}$ normal to the t_{P}-axis in $\mathbb{R}^4_{\mathrm{P}}$ is introduced, and then σ_t in $\mathbb{R}^4_{\mathrm{M}}$ is defined as a three-dimensional hypersurface mapped from $\sigma_{\mathrm{P}t}$ under the function in Eqs. (2.1)−(2.3). For $x_\mu, x'_\mu \in \mathbb{R}^4_{\mathrm{M}}$ on the space-like hypersurface σ_t at the point x_μ, the following relation [5, 9, 10, 15] is satisfied

$$(t' - t)^2 \leq (x'_i - x_i)^2. \tag{2.4}$$

2.3 Poincaré invariance/covariance of plane waves

In this subsection, Poincaré invariant/covariant plane waves are considered. Using four-momentum p_μ, a plane waves is expressed as a function of an inner product $p_\mu x_\mu$ in the form defined by Eq. (2.8) below. The plane wave is used for a Fourier expansion without and with the cutoff in the momentum space, as mentioned in Subsec. 2.1.

Lemma 2.5. *For $x_{(1)}, x_{(2)}, x'_{(1)}, x'_{(2)} \in \mathbb{R}^4_{\mathrm{E}}$, let $L_{\mu\nu}$ be a matrix that represents a Lorentz transformation. For vectors $x_{(2)} - x_{(1)}$ and $x'_{(2)} - x'_{(1)}$, the following Lorentz transformation is considered*

$$x_{(2)} - x_{(1)} = x_{(2)\nu} - x_{(1)\nu} = L_{\nu\mu}(x'_{(2)\mu} - x'_{(1)\mu}). \tag{2.5}$$

Then for $x_{(1)}, x_{(2)}, x_{(3)}, x_{(4)} \in \mathbb{R}^4_{\mathrm{E}}$ as well as vectors $x_{(2)} - x_{(1)}$ and $x_{(4)} - x_{(3)}$, the scalar inner product of these vectors is Lorentz covariant/invariant:

$$\begin{aligned} P_{\mathrm{I}(43,21)} &:= (x_{(4)} - x_{(3)}) \cdot (x_{(2)} - x_{(1)}) \\ &= (x'_{(4)\mu} - x'_{(3)\mu})(x'_{(2)\mu} - x'_{(1)\mu}). \end{aligned} \tag{2.6}$$

Proof. The above lemma is simply shown as

$$\begin{aligned} P_{\mathrm{I}(43,21)} &= L_{\nu\eta}(x'_{(4)\eta} - x'_{(3)\eta}) L_{\nu\mu}(x'_{(2)\mu} - x'_{(1)\mu}) \\ &= L^{-1}_{\eta\nu}(x'_{(4)\eta} - x'_{(3)\eta}) L_{\nu\mu}(x'_{(2)\mu} - x'_{(1)\mu}) \\ &= \delta_{\eta\mu}(x'_{(4)\eta} - x'_{(3)\eta})(x'_{(2)\mu} - x'_{(1)\mu}), \end{aligned} \tag{2.7}$$

where $\delta_{\eta\mu}$ is Kronecker's delta. □

Lemma 2.6. *For $x_{(1)}, x_{(2)} \in \mathbb{R}^4_{\mathrm{E}}$ as well as a constant four-vector a_μ, let $x_{(1)\mu} = x'_{(1)\mu} + a_\mu$ and $x_{(2)\mu} = x'_{(2)\mu} + a_\mu$ be translations contained in the Poincaré transformation. Using the quantities in Lemma 2.5, and considering that $x_{(2)\mu} - x_{(1)\mu} = x'_{(2)\mu} - x'_{(1)\mu}$ and $P_{\mathrm{I}(43,21)}$ in Eq. (2.6) are Poincaré covariant/invariant, it follows that a plane wave*

$$\chi_p(x) := \exp(i p_\mu x_\mu), \tag{2.8}$$

is a scalar function for $x_\mu \in \mathbb{R}^4_{\mathrm{E}}$ and a four-vector p_μ.

Proof. The above lemma is simply because in Lemma 2.6, the quantity x_μ can be regarded as a vector for $x_{(2)\mu} = x_\mu$ and $x_{(1)\mu} = (0,0,0,0)$. Here, $x_{(4)\mu} - x_{(3)\mu}$ is regarded as p_μ in Eq. (2.8). □

2.4 The Poincaré invariant/covariant first cutoff at the energy lower than the second cutoff energy

Here, let us introduce two Poincaré invariant/covariant cutoffs. Physically, the first cutoff at the energy lower than the second cut at the Planck energy is effective at removing the large quadratic self-energies associated with the Higgs field. By analogy with solid state physics as described in Subsec. 2.1, the first cutoff is achieved owing to the division of the spacetime continuum into spacetime elements. In this case, the wave length is longer than the size of the spacetime element. As a result, continuous momentums of the usual Yang-Mills field are restricted to the regime within the first cutoff energy. On the other hand, it is stated in the literature [26] that when a function on the Euclidean spacetime has the same local periodicity as that of the local alignment of the spacetime elements, the function has Fourier components of the discrete momentum; and this discrete momentum is the reciprocal cutoff four-momentum multiplied by integers Here, the present theory introduces an effective compensation vector field in the form of this periodic function with discrete momentums. By interactions with the effective compensation field, quantum Yang-Mills gluon particles are only rarely excited beyond the first cutoff energy. Furthermore, above the first cutoff energy, the second cutoff is introduced, which corresponds physically to the cutoff at the Planck energy. This second cutoff is due to the fact that in intermediate states above the Plank energy, unphysical losses of probabilities associated with particles occur by the absorption of black holes.

Definition 2.7. Let $a_r \in \mathbb{R}$ be the separation between each center of the nearest-neighbor space-time elements, whose alignment is locally periodic without any long-range order. Four-momentums are limited within in the following first momentum zone by the cutoff

$$|\underline{k}| := |\underline{k}_\mu| = |(\underline{k}_\mu \underline{k}_\mu)^{1/2}| \leq \pi/a_r, \tag{2.9}$$

and plane waves with the cutoff are expressed as

$$\underline{\chi}_{\underline{k}} := \exp(\underline{k}_\mu x_\mu). \tag{2.10}$$

For the quantities whose momentums are limited within the cutoff energy, an underline under the symbol(s) such as $\underline{\chi}$ and $\underline{k}$ is added, although this underline is not always denoted explicitly.

It is noted that the underline is added explicitly only if it is necessary to distinguish between the quantities with and without the cutoff. The momentums beyond the second cutoff energy at the Planck energy are also cut off.

Definition 2.8. Let a_r be the separation between spacetime elements in the above Definition 2.7. Let $x_p = x_{p\mu} = (x_{p0}, x_{p1}, x_{p2}, x_{p3})$ be a center of the space-time element. For integers n_0, n_1, n_2, n_3, quantities $G_{\tilde{n}} := G_{\mu\tilde{n}} := G_{\mu\tilde{n}p}$ at x_p are defined by

$$\begin{aligned} G_{0\tilde{n}p} &:= n_0(2\pi/a_r)e_{0p}, \quad G_{1\tilde{n}p} := n_1(2\pi/a_r)e_{1p}, \\ G_{2\tilde{n}p} &:= n_2(2\pi/a_r)e_{2p}, \quad G_{3\tilde{n}p} := n_3(2\pi/a_r)e_{3p}. \end{aligned} \tag{2.11}$$

Here, $e_{\mu p}$ at x_p are unit vectors, which are mutually orthogonal, and are oriented in arbitrary directions.

Lemma 2.9. *On the spacetime continuum $\mathbb{R}^4_{\mathrm{E}}$, which is divided into spacetime elements aligned with a local periodicity without long-range order, let us consider a function $u(x)$, which has the same periodicity as the aligned spacetime elements. Then $u(x)$ is expanded in the form of the following Fourier series in terms of $G_{\mu\tilde{n}}$ in Definition 2.8*

$$u(x) = \sum_{G_{\tilde{n}}} u_{G_{\tilde{n}}} \exp(iG_{\mu\tilde{n}} x_\mu) \quad \textit{with} \quad G_{\tilde{n}} := G_{\mu\tilde{n}}. \tag{2.12}$$

The proof of the above lemma is given in Chap. 1 of the literature [26].

Here, the following effective compensation vector field is newly introduced.

Definition 2.10. Using the four-momentums $G_{\nu\tilde{n}}$ in Definition 2.8, let us introduce the following compensation vector field, which has the same local periodicity as the periodically aligned spacetime elements that are obtained by dividing the spacetime continuum $\mathbb{R}^4_{\mathrm{E}}$,

$$A_{(\mathrm{EC})\mu}(x) := \sum_{G_{\tilde{n}}} A_{(\mathrm{EC})\mu G_{\tilde{n}}} \exp(iG_{\nu\tilde{n}} x_\nu), \tag{2.13}$$

where $A_{(\mathrm{EC})\mu G_{\tilde{n}}}$ are expansion coefficients.

Definition 2.11. Only via interactions with the effective compensation field of discrete momentum in Definition 2.10, a particle described by a usual plane wave is rarely excited by an additional discrete momentum. The plane wave then has the following form

$$\begin{aligned} \chi_{\underline{k}G_{\tilde{n}}}(x) &:= \sum_{G_{\tilde{n}}} \exp[i(\underline{k}_\mu + G_{\mu\tilde{n}})x_\mu] \\ &= \sum_{G\tilde{n}} \exp(iP_\mu x_\mu). \end{aligned} \tag{2.14}$$

In the above equation the quantity $\underline{k}_\mu$ is the four-momentum in the first momentum zone given by Eq. (2.9) using a_r in Definition 2.7. Additionally, the momentum P_μ of the quantum particle becomes $P_\mu = \underline{k}_\mu + G_{\mu\tilde{n}p}$.

In the following, a special unitary group is considered as an example of a compact and simple group. This treatment does not involve any loss of generality of the obtained results for compact and simple groups.

Definition 2.12. For a meaningful natural number N, let us here consider a special unitary group SU(2) or a more extended SU(N) represented by matrices whose dimension is $N_{\mathrm{D}} = N^2 - 1$, as an example of a compact and simple group. Let T^a be traceless Hermitian matrices defined as generators of the group, and generate the commutators

$$[T^a, T^b] := T^a T^b - T^b T^a = \sum_c i f^{abc} T^c, \tag{2.15}$$

where the indices a, b and c run from 1 to N_{D}, and f^{abc} are totally antisymmetric structure constants. It is noted that the summation of the quantities over these indices is explicitly denoted. Using parameters $\omega_a \in \mathbb{R}$, the SU(N) group is represented by the following special unitary matrix

$$U(x) := \exp[ig \sum_a \omega_a(x) T^a], \tag{2.16}$$

where g denotes a coupling constant. For $x_\nu \in \mathbb{R}^4$, a Yang-Mills field $A_\mu(x)$ has the form of

$$A_\mu(x) = A_\mu(x_\nu) = A_\mu(t, \mathbf{x}) := \sum_a A^a_\mu(x) T^a. \tag{2.17}$$

It is also noted that each component $A^a_\mu(x)$ of the Yang-Mills field $A_\mu(x)$ is multiplied by the generator T^a of the group.

Definition 2.13. Let $\chi_p(x)$ be plane waves denoted by

$$\chi_p(x) = \exp(ip_\mu x_\mu), \tag{2.18}$$

where the momentum p_μ takes any value in the 4D momentum space. Then, using Definition 2.12, the quantum Yang-Mills field is expressed in the Fourier expansion form

$$A_\mu(x) = \sum_a \sum_p A^a_{\mu p} T^a \chi_p(x). \tag{2.19}$$

Whereas, using the notation

$$\underline{\chi}_{\underline{k}}(x) = \exp(i\underline{k}_\mu x_\mu), \qquad \text{with } |\underline{k}| \leq \pi/a_r, \tag{2.20}$$

as well as Definition 2.12, the quantum Yang-Mills field, whose momentums are restricted to the first momentum zone $|\underline{k}| \leq \pi/a_r$ due to the low-energy (first) cutoff in the Poincaré covariant form, is represented as follows:

$$\underline{A}_\mu(x) := \sum_a \sum_{\underline{k}} \underline{A}^a_{\mu\underline{k}} T^a \underline{\chi}_{\underline{k}}(x) \qquad \text{with } |\underline{k}| \leq \pi/a_r. \tag{2.21}$$

As previously mentioned, π/a_r is the cutoff value (for the low-energy cutoff) in the energy-momentum space, where a_r denotes the distance between the nearest-neighbor spacetime elements obtained by dividing the spacetime continuum.

It is noted that since the expansion functions $\chi_k(x)$ are scalar plane waves in Subsec. 2.3, the individual coefficients $A^a_{\mu k} T^a$ transform as a vector and a tensor under a transformation of the Lorentz group and SU(N) (and similar groups), respectively.

Definition 2.14. According mainly to the notations in the literature [2] for the Yang-Mills field, the Lagrangian density of the pure Yang-Mills field is defined by

$$\begin{aligned} \mathscr{L}_{\mathrm{YM}} &:= -\frac{2}{4}\mathrm{Tr}[F_{\mu\nu}(x)F_{\mu\nu}(x)] \\ &= -\frac{1}{4}\sum_a F^a_{\mu\nu}(x)F^a_{\mu\nu}(x), \end{aligned} \tag{2.22}$$

where Tr is trace of the matrix, and

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - g\sum_{b,c} f^{abc} A^b_\mu(x) A^c_\nu(x). \tag{2.23}$$

Although this paper focuses on pure Yang-Mills fields, it briefly summarizes that the Yang-Mills field is a gauge field. A fermion field $\psi(x)$ comprises components $\psi^a(x)$, and the Lagrangian density of fermions with a mass of nearly zero is denoted as

$$\mathscr{L}_{\mathrm{F}} := -\psi^* \gamma_4 \gamma_\mu \partial_\mu \psi(x), \tag{2.24}$$

where γ_μ are Dirac matrices. For the gauge transformation of the fermion field

$$\psi(x) \to U(x)\psi(x), \tag{2.25}$$

the covariant derivative is defined by

$$D_{\mu\nu} := \partial_\mu + igA_\mu(x). \tag{2.26}$$

Then, the gauge transformation of the Yang-Mills field is expressed as

$$A_\mu(x) \to U(x)A_\mu(x)U^{-1}(x) - \frac{i}{g}U(x)\partial_\mu U^{-1}(x). \tag{2.27}$$

2.5 Vacuum expectation value for a product of operators with the cutoff

As aforementioned in Subsec. 2.1, this subsection first describes the Feynman diagram method. Subsequently, corresponding physically to an vacuum expectation value for a product of operators, an expectation value for a quantity is presented, in the case without and with the cutoff. The established Feynman diagram method has been described in detail in many literatures [2, 4–6, 12–15], and this paper briefly summaries this method. The integral for the expectation value in the Minkowski momentum space is rewritten in the Euclidean momentum space using the residue theorem on a complex plane.

A state vector $|\Psi>$, which describes a quantum Yang-Mills system, transit to another state $|\Psi'>$ by multiplication by a matrix $U_{\rm I}$, as follows:

$$|\Psi'>=U_{\rm I}|\Psi> . \tag{2.28}$$

The probability (transition) amplitude projected to the vector $<\Psi''|$ is given by

$$<\Psi''|\Psi'>=<\Psi''|U_{\rm I}|\Psi> . \tag{2.29}$$

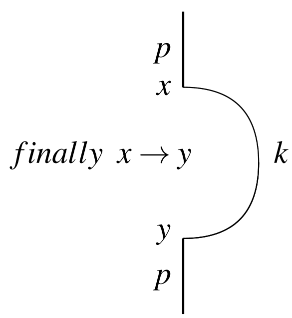

Figure 1: A Feynman diagram for a self-interaction of a field.

The above $U_{\rm I}$ is expanded by a power series of a coupling constant g, and the general term of the series is depicted using the Feynman diagram, as shown in Fig. 1. This figure is an example of a graph, which describes a self-interaction of the field, with final setting of $x \to y$. In this interaction, a quantum particle named gluon with an initial four-momentum p annihilates at the intersection y of two (straight or curved) lines, followed by the creation of a particle with momentum k. The particle with k is annihilated at the intersection x, and a particle with the momentum p is created. The straight or curved line indicates the field-particle propagation without momentum change. In canonical quantization, at the intersection of straight/curved lines, particles with each momentum are created/annihilated due to creation/annihilation operators and the associated coupling constant. The intersection is a location where the momentum changes. It is known that path integral quantization without creation and annihilation operators is equivalent to canonical quantization and provides an identical expectation value. Finally, by putting $x \to y$, this process becomes the case, in which emitted particle is reabsorbed at the same intersection, In quantum mechanics, the absolute momentum $|k|$ in this intermediated state takes any value from zero to infinity. The process between the intersections x and y is the interaction in quantum theory. The freedom of the momentum k for this self-interaction in the intermediate state is excessively large; therefore the integral over the momentum k results in an ultraviolet divergence. The solution in this paper prevents the ultraviolet divergence by the Poincaré invariant/covariant cutoff, which removes contributions from high energies beyond the cutoff energy, and enables us to treat fields without ultraviolet divergence.

Before describing the expression depicted by the Feynman diagram in Fig. 1, the interaction of one component of the Yang-Mills fields is roughly written as

$$\begin{aligned}\approx \delta_{\mu\mu'} g^2 \int d^3k \ \Big\{ \frac{1}{(2\pi)^3 2k_0} a^{*a}_{\mu'} \exp(-ipx) \theta(x_0 - y_0) \exp[ik(x-y)] \\ \times < 0 | a^a_\mu(\mathbf{k}) a^{*a}_\mu(\mathbf{k}) | 0 > a^a_{\mu'} \exp(ipx) \Big\},\end{aligned} \tag{2.30}$$

where $a^{*a}_\mu(\mathbf{k})$ and $a^a_\mu(\mathbf{k})$ are creation and annihilation operators of a quantum particle of the Yang-Mills field for the component a. Here,

$$\theta(t) := \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}, \tag{2.31}$$

with $\theta(0)$:=1/2. For the rough Eq. (2.30), the strict expression is derived below, noting that the quantities with the repeated group index a are not summed. The fields without the cutoff have any four-momentums, including those beyond the Planck energy. The Yang-Mills field $A_\mu(x) = T^a A^a(x)$ without the cutoff is expressed in the form of a Fourier expansion

$$A^a_\mu(x) = \int d^3k \ \frac{1}{\sqrt{(2\pi)^3 2k_0}} \left[a^a_\mu(\mathbf{k}) \exp(ik_\mu x_\mu) + a^{*a}_\mu(\mathbf{k}) \exp(-ik_\mu x_\mu) \right], \tag{2.32}$$

where k_μ is the energy-momentum, and $\mathbf{k}$ is the three-dimensional momentum. The field without cutoff has some difficulties. In the quantum field scheme, an interacting field exchanges an quantum particle with any momentum in the intermediate state. Unfortunately, it is known that a particle with the energy higher than the Planck energy becomes a black hole. Then, the particle does not return to the original particle from the black hole. Moreover, self-interactions associated with the Higgs particle unphysically enhance the self-energy to a huge value, unless the cutoff energy is much lower than the Planck energy. In this paper, the low-energy (first) cutoff is introduced consistently with the high-energy (second) cutoff at the Planck scale. The field with the four-momentum $\underline{k}_\mu$ within the cutoff energy is denoted by $\underline{A}_\mu(x) = T^a \underline{A}^a(x)$. For $\underline{A}_\mu(x)$, the field expression corresponding to Eq. (2.32) becomes

$$\underline{A}^a_\mu(x) = \int d^3\underline{k} \ \frac{1}{\sqrt{(2\pi)^3 2\underline{k}_0}} \left[a^a_\mu(\underline{\mathbf{k}}) \exp(i\underline{k}_\mu x_\mu) + a^{*a}_\mu(\underline{\mathbf{k}}) \exp(-i\underline{k}_\mu x_\mu) \right]. \tag{2.33}$$

Hereafter, only when necessary, the underlined symbols such as $\underline{k}_\mu$ and $\underline{A}_\mu(x)$ with the cutoff are explicitly indicated, and most field quantities with the energy cutoff are not distinguished from those without the cutoff. Using the symbol $[A,B] = AB - BA$, the creation and annihilation operators satisfy

$$\begin{aligned}&[a^a_\mu(\mathbf{k}), a^{*b}_\nu(\mathbf{k}')] = \delta_{ab}\delta_{\mu\nu}\delta_{\mathbf{k}\mathbf{k}'}, \quad [a^a_\mu(\mathbf{k}), a^b_\nu(\mathbf{k}')] = 0, \quad [a^{*a}_\mu(\mathbf{k}), a^{*b}_\nu(\mathbf{k}')] = 0, \\ &a^a_\mu(\mathbf{k})|0 >= 0.\end{aligned} \tag{2.34}$$

Additionally, T product, which is the time-ordered product of operators, is defined by

$$\mathrm{T}A^a_\mu(x)A^b_\nu(y) = \theta(x_0-y_0)A^a_\mu(x)A^b_\nu(y) + \theta(y_0-x_0)A^b_\nu(y)A^a_\mu(x), \tag{2.35}$$

using Eq. (2.31).

For a positive natural number n, the coordinates x, y are generalized to $x_{(1)},\cdots,x_{(n)}$, and the abbreviated quantities

$$b(x) := A^a_\mu(x), \qquad b(y) := A^b_\nu(y), \tag{2.36}$$

are generalized to $b(x_{(1)}),\cdots,b(x_{(n)})$. For a positive natural number j, it is known that U_I in Eq. (2.28) is expanded into a power series with respect to the coupling constant g, and each term is depicted by a Feynman graph. This term is written in the following form, which is further multiplied by a power of g and structure constants of the group,

$$\sum_{\mathrm{comb}} [\mathrm{N}b(x'_{(1)})\cdots b(x'_{(j)})] < 0|\mathrm{T}b(x'_{(j+1)})\cdots b(x'_{(n)})|0>, \tag{2.37}$$

where $\sum_{\mathrm{comb}}$ denotes the sum over all combinations of $x_{(1)},\cdots,x_{(n)}$. The quantity $\mathrm{N}b(x'_{(1)})\cdots b(x'_{(j)})$ is defined as N product in which creation operators $a^{*a}_\mu(\mathbf{k})$ and annihilation operators $a^a_\mu(\mathbf{k})$ are placed on the left and right, respectively. For odd n, the above expectation value $< 0|\mathrm{T}b(x'_{(j+1)})\cdots b(x'_{(n)})|0>$, which physically corresponds to the vacuum expectation value, vanishes, while for even n, the value is rewritten as

$$\begin{aligned} &< 0|\mathrm{T}b(x'_{(j+1)})b(x_{(j+2)})\cdots b(x'_{(n)})|0> \\ =& \sum_{k_1<k_2,k_3<k_4,\cdots} < 0|\mathrm{T}b(x'_{(k_1)})b(x'_{(k_2)})|0> \cdots < 0|\mathrm{T}b(x'_{(k_{n-1})})b(x'_{(k_n)})|0>, \end{aligned} \tag{2.38}$$

where $k_1,k_2,\cdots,k_n$ are permutations of $j+1, j+2,\cdots,n$. Using Eqs. (2.32)$-$(2.35) denoted by the original notations, such as $A^a_\mu(x)$ for $b(x'_{(k_1)})$ in Eq.(2.36), the above (vacuum) expectation value multiplied by g^2 is given by

$$\begin{aligned} & g^2 < 0|\mathrm{T}A^a_\mu(x)A^b_\nu(y)|0> \\ =\;& g^2\delta_{ab}\int d^3k d^3q \;\Bigg\{ \frac{1}{\sqrt{(2\pi)^3 2k_0}}\frac{1}{\sqrt{(2\pi)^3 2q_0}} \\ & \times[\theta(x_0-y_0)\exp(ikx-iqy) < a^a_\mu(\mathbf{k})a^{*b}_\nu(\mathbf{q})|0> \\ & +\theta(y_0-x_0)\exp(-ikx+iqy) < a^b_\nu(\mathbf{q})a^{*a}_\mu(\mathbf{k})|0>]\Bigg\} \\ =\;& g^2\delta_{ab}\delta_{\mu\nu}\int d^3k \;\Bigg\{\frac{1}{(2\pi)^3 2k_0} \\ & \times[\theta(x_0-y_0)\exp(ik(x-y)) + \theta(y_0-x_0)\exp(-ik(x-y))]\Bigg\}. \end{aligned} \tag{2.39}$$

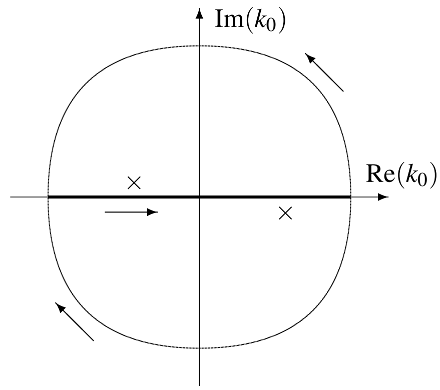

Figure 2: Contour of the integral for a (vacuum) expectation value using the residue theorem.

Now, Eq. (2.39) is expressed in the following integral with the help of the residue theorem of complex analysis:

$$\begin{aligned} g^2 \quad & < 0|\mathrm{T}A^a_\mu(x)A^b_\nu(x)|0 > \\ = \quad & \delta_{ab}\delta_{\mu\nu}g^2 \int d^4k \ \frac{1}{i(2\pi)^4}\frac{1}{k^2+(m_\varepsilon)^2-i\varepsilon}\exp[ik(x-y)], \end{aligned} \tag{2.40}$$

where ε is an infinitesimal value, and the limits of $m_\varepsilon \to 0$, $\varepsilon \to 0$ are taken after calculations. Let us extend the real number k_0 to the complex number shown in Fig. 2. In the integral $\int_{-\infty}^{0} dk_0(\cdots)$ on the real axis, the pole indicated by $\times$ that lies at $-\sqrt{\mathbf{k}^2+(m_\varepsilon)^2-i\varepsilon}= -\sqrt{\mathbf{k}^2+(m_\varepsilon)^2}+i\varepsilon'$ infinitesimally above the real axis. Here, ε' is also an infinitesimal quantity, which is taken as $\varepsilon' \to$ after calculations. Similarly, for the integral $\int_0^{+\infty} dk_0(\cdots)$ on the real axis, the pole denoted as $\times$ that lies at $+\sqrt{\mathbf{k}^2+(m_\varepsilon)^2-i\varepsilon}= +\sqrt{\mathbf{k}^2+(m_\varepsilon)^2}-i\varepsilon'$ infinitesimally below the real axis. For the field without the cutoff for $x_0-y_0>0$, we add the vanishing integral on the lower half circle infinitely distant from the origin as shown in Fig. 2. For $x_0-y_0<0$, we also add the integral on the upper half circle. Based on the residue theorem, Eq. (2.40) reproduces Eq. (2.39). For the field with the cutoff, the outcome is the same as that for the case without the cutoff, because the cutoff leads to vanishing of the function of the momentum in the region that is sufficiently distant from the origin.

Next, the integral above is transformed to the Euclidean integral using the residue theorem. As mentioned below, the integral path is rotated by $\pi/2$, as shown in Fig. 3. For k_0 on the complex plane, the integral $\int_{-\infty}^{+\infty} dk_0(\cdots)$ on the real axis is extended by adding the vanishing integrals $\oint_0^{\pi/2}$ (which is rather symbolic) and $\oint_{-\pi/2}^{-\pi}$ on an infinitely large radius circle. Furthermore, we also add the integral on the imaginary axis $\int_{+i\infty}^{-i\infty} dk_0$. Then, based on the residue theorem, the integral on the real k_0 axis becomes that over the imaginary ik_0 axis with k_0 being real (k_0 is the limited meaning used here only). The quantity $-(k_0)^2$ in the integrand also becomes $(k_0)^2$ that is equivalent to the Euclidean metric. This implies that using the contour rotation, the integral with the Minkowski metric is transformed to that with the Euclidean metric. From Eq. (2.40), the Euclidean integral in the limit of $m_\varepsilon \to 0$, $\varepsilon \to 0$, $\varepsilon' \to 0$ yields

$$g^2 < 0|\mathrm{T}A_\mu^a(x)A_\nu^b(x)|0 >= g^2\delta_{ab}\delta_{\mu\nu}\int d^4k \ \frac{1}{(2\pi)^4}\frac{1}{k^2}\exp[ik(x-y)]. \tag{2.41}$$

The above (vacuum) expectation value has the form of a Fourier series as a function of momentum in the Euclidean space. It is known that the transition amplitude in Eq. (2.29) contains self-interaction quantities expressed by the integral over momentums for the individual (vacuum) expectation value. An example of the self-interaction is depicted by a Feynman graph in Fig. 1.

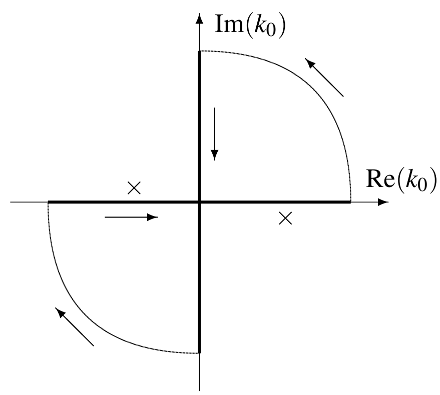

Figure 3: Contour for the energy integral on the real axis in Minkowski space and rotation to that on the imaginary axis.

Lemma 2.15. *A self-interaction quantity contained in the transition amplitude in Eq. (2.29) diverges in the case without the cutoff, while the self-interaction quantity has a finite value in the case with the cutoff.*

Proof. A self-interaction quantity depicted by such as a graph in Fig. 1 has the form

$$\frac{1}{k^2}. \tag{2.42}$$

In the Euclidean-space integral, the following relation is used

$$d^4k = dk \ 2\pi^2k^3 = d(k^2) \ \pi^2(k^2). \tag{2.43}$$

For the field without the cutoff, the integral in Eq. (2.41) with respect to k^2 over all momentum values amounts to

$$\int d(k^2)k^2\frac{1}{k^2} = \infty, \tag{2.44}$$

which is referred to as the ultraviolet divergence.

Whereas, for the field with the cutoff at the rotation-invariant Λ_{CO}, the momentum in the region within the cutoff value is here explicitly denoted as $\underline{k}$ in Eq. (2.33) instead of k. In this case, the above integral is derived to give

$$\int_0^{(\Lambda_{\mathrm{CO}})^2} d(\underline{k}^2)\underline{k}^2 \frac{1}{\underline{k}^2} = (\Lambda_{\mathrm{CO}})^2, \tag{2.45}$$

which is proportional to the finite $(\Lambda_{\mathrm{CO}})^2$ and the ultraviolet divergence is removed.

□

The above integral for the multi-component gauge field is multiplied by the antisymmetric structure constants. If the absolute value of one term has the same value as that of another term with the opposite sign, these terms are cancelled.

2.6 Relationship between vacuum expectation values of operator products and tempered distributions

This subsection describes the relationship between (vacuum) expectation values in the Minkowski spacetime and tempered distributions. The procedure in the present Sec. 2 finally derives Theorem 2.20, which presents a solution to the Yang-Mills existence problem. A mathematical expectation value for a product of operators is here modified using the cutoff at an energy E_{co}. For a proof of the lemma below, one can refer to the literatures [4, 5]. Because vector fields are expanded in terms of scalar plane fields, the essence of vector fields is obtained from the scalar fields.

Definition 2.16. For $x \in \mathbb{R}^4_{\mathrm{M}}$, let $\phi(x)$ be a massless scalar field, which satisfies $(-\partial_t\partial_t + \partial_i\partial_i)\phi(t,\mathbf{r}) = 0$ in the no interaction case. In the interaction picture, the expectation value for a product of two-point operators $\phi(x)\phi(y)$ is denoted by $< 0|\mathrm{T}\phi(x)\phi(y)|0 >$, where $\mathrm{T}\phi(x)\phi(y)$ is a time-ordered product (T product) of operators defined by

$$\mathrm{T}\phi(x)\phi(y) := \begin{cases} \phi(x)\phi(y) & \text{for } x_0 > y_0 \\ \phi(y)\phi(x) & \text{for } x_0 < y_0 \end{cases}. \tag{2.46}$$

Lemma 2.17. *According to the literature [5], the expectation value $< 0|\mathrm{T}\phi(x)\phi(y)|0 >$ in Definition 2.16 is given by*

$$< 0|\mathrm{T}\phi(x)\phi(y)|0 >= \begin{cases} i\frac{-i}{2(2\pi)^3} \int\limits_{k_0>0} d^3k \\ \frac{\exp[ik_\mu(x_\mu - y_\mu)]}{k_0} & \mathit{for}\ x_0 > y_0 \\ -i\frac{i}{2(2\pi)^3} \int\limits_{k_0>0} d^3k \ \frac{\exp[-ik_\mu(x_\mu - y_\mu)]}{k_0} & \mathit{for}\ x_0 < y_0 \end{cases}. \tag{2.47}$$

The detailed proof of the above relationship is described in the literature [5] by Schweber. A slightly different representation and derivation procedure for general fields were presented in Eq. (2.39).

Lemma 2.18. *One of the above integrals yields*

$$\frac{i}{2(2\pi)^3}\int_{k_0>0} d^3k \ \frac{\exp(ik_\mu x_\mu)}{k_0} = \frac{4\pi}{r}\delta(r-t). \tag{2.48}$$

Proof. As written in the literatures by Tomonaga and coworkers (N. Fukuda, H. Fukuda, and K. Sawada) in Japanese as well as by Heitler [4], the above integral is reduced to

$$\begin{aligned}
&\frac{i}{2(2\pi)^3}\int_{k_0>0} d^3k \ \frac{\exp(ik_\mu x_\mu)}{k_0} \\
&= \frac{2\pi i}{(2\pi)^3}\int_0^\infty dk \ \frac{k}{2}\int_0^\pi d\theta \ \ \sin\theta \exp(ikr\cos\theta - ikt) \\
&= \frac{4\pi}{(2\pi)^3 r}\int_0^\infty dk \ \ \sin kr \sin kt = \frac{1}{8\pi^2 r}\int_{-\infty}^\infty dk \ \ \exp[ik(r-t)] \\
&= \frac{1}{4\pi r}\delta(r-t).
\end{aligned} \tag{2.49}$$

□

The above physical δ-function corresponds to a mathematical tempered distribution, and indicates a signal propagation. However, the above integral causes ultraviolet divergences. The theoretical framework of this paper removes ultraviolet divergences using the first and second cutoffs. For the above Eq. (2.49), the below Eq. (2.56) is to be given as a simple example of the cutoff.

Corresponding physically to singular invariant functions in field theory, mathematical tempered distributions, which are included in Schwartz distributions, are usually utilized with the benefit that Fourier transformations are available. A simple example of the physical $\delta(x)$-function, which was introduced by Dirac, is redefined mathematically as one of tempered distributions. Let us consider a test function f_{T}: $\mathbb{R} \to \mathbb{R}$. This is generally written by f_{T}: $U \to \mathbb{R}$ with U being an open set in $\mathbb{R}$; and $\mathbb{R}$ is generalized to $\mathbb{R}^4$. For $\alpha, \alpha^\mu, \beta, \in \mathbb{N}$ with $\mathbb{N}$ being a set of non-negative natural numbers in this case, higher-order partial derivative is denoted by $D^\alpha f_{\mathrm{T}}(x) := \partial_x^\alpha f_{\mathrm{T}}(x)$. The differentiation is generally expressed as $D^\alpha := \partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ with $\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$, and the derivative satisfies

$$\sup_{x\in\mathbb{R}} |x^\beta D^\alpha f_{\mathrm{T}}(x)| < \infty, \tag{2.50}$$

where the symbol sup denotes the supremum. Then a set of the test functions f_{T} which have the above properties is denoted by D. The Dirac delta function $\delta(x) \in D'$, with D' being a set of the tempered distributions, is introduced using the following scalar inner product with $f_{\mathrm{T}} \in D$

$$\int dx \ \ \delta(x) f_{\mathrm{T}}(x) := f_{\mathrm{T}}(0). \tag{2.51}$$

More generally, $s(x) \in D'$ is introduced using the following scalar inner product with the test function $f_{\rm T} \in D$

$$< s, f_{\rm T} >:= \int dx \;\; s(x) f_{\rm T}(x). \tag{2.52}$$

As is known, derivatives as well as the Fourier and inverse transforms are mathematically defined similarly. The Fourier transform of the δ-function yields

$$\frac{1}{\sqrt{2\pi}} \int dx \;\; \exp(-ipx)\delta(x) = \frac{1}{\sqrt{2\pi}}. \tag{2.53}$$

The inverse Fourier transform of the above function is then

$$\frac{1}{\sqrt{2\pi}} \int dp \;\; \exp(ipx) \frac{1}{\sqrt{2\pi}} = \delta(x). \tag{2.54}$$

Lemma 2.19. *Let $E_{\rm co}$ be the cutoff energy. For simplicity, with the cutoff of solely energies for the integral $\int_{-\infty}^{\infty} dk \;\; \exp[ik(r-t)]$ in Eq. (2.49) is regularlized to give*

$$\int_{-E_{\rm co}}^{E_{\rm co}} dk \;\; \exp[ik(r-t)] = \frac{2\sin[E_{\rm co}(r-t)]}{r-t}. \tag{2.55}$$

Proof. The integral above results in

$$\begin{aligned} &\int_{-E_{\rm co}}^{E_{\rm co}} dk \;\; \exp[ik(r-t)] \\ &= \frac{1}{i(r-t)} \{\exp[iE_{\rm co}(r-t)] - \exp[-iE_{\rm co}(r-t)]\} \\ &= \frac{2\sin[E_{\rm co}(r-t)]}{r-t}. \end{aligned} \tag{2.56}$$

□

It is noted that the numerator of the above integral is written in the form of $2E_{\rm co}(r-t)+$ higher-order terms, and the integral has a finite value at $r-t=0$. When the scalar field has a mass $\mu_{\rm m}$, the following integral in Eq. (2.49)

$$\int_0^{\infty} dk \;\; \sin kr \sin kt \tag{2.57}$$

is replaced by

$$\frac{\partial}{\partial_r} \int_0^{\infty} dk \;\; \frac{1}{\sqrt{k^2+\mu_{\rm m}^2}} \cos kr \sin k_0 t, \tag{2.58}$$

according to the literature [5].

The procedure without ultraviolet divergences in this Sec. 2 leads to the following theorem.

Theorem 2.20. *By introducing two cutoffs with Poincaré covariance/invariance, the formalism in this paper defines pure Yang-Mills fields without ultraviolet divergences. Since vector fields are expanded in terms of scalar plane waves, the expansion coefficients of the fields are transformed as vectors under a Lorentz transformation. At the same time, each component of the Yang-Mills and its generalized fields is multiplied by the generator of the $SU(N)$ group, which is an example of compact and simple groups. Each of these fields constructs a covariant derivative as a gauge field with fermion fields, thereby forming an invariant/covariant system under the gauge transformation via a unitary matrix in terms of the group generators. This is an answer to the Yang-Mills existence problem of Millennium problem.*

It is noted that by removing the high-energy contributions, the present formalism is well-defined and consistent with the corresponding properties of the axiomatic formalism [5] by Streater & Wightman (SW) or equivalently Osterwalder & Schrader (OS), because these axioms are constructed based on physical fields such as Yang-Mills fields.

3 A solution to the mass gap problem for pure Yang-Mills fields

3.1 Aim and outline of Sec. 3

This section presents a solution to the mass gap problem. This solution provides a property that although the Yang-Mills fields are gauge fields, these fields are not long ranged because of a non-vanishing mass. Furthermore, it is shown in a physical meaning that a pair of a quantum particle (gluon) and an antiparticle of the Yang-Mills field is strongly bound, making the pair separation difficult. This binding energy of the pair is equal to a non-vanishing mass of the field. In the case of quarks mediated by the Yang-Mills field, it is experimentally shown by the Regge trajectory that the long-range potential associated with the force between a quark and an antiquark is linear. Whereas, the short-range interaction in the same quark case reveals that the particles moves rather freely under the Coulomb potential. Based on these experimental results, this section shows that the potential between a gluon particle and an antiparticle of a pure Yang-Mills field is a combination of linear and Coulomb potentials, similarly to the quark case. The Feynman path integral, a field quantization formalism, calculates an expectation value of a quantity, by summing probability values over all field configurations associated with the individual weight of the action functional. The action is expressed by an integral of the Lagrangian density on the spacetime continuum. This quantization is regarded as fluctuations from a stationary state derived from a vanishing variation in analytic dynamics. The stationary state is described by a differential field equation, which provides a solution of a classical field. This approach to a non-perturbative stationary state is similar to the intermediate-coupling theory [31] formulated by Tomonaga and coworkers. The solution resembles to a Laurant series containing both a term $1/z$ and the remainder expressed by a power series in complex analysis, where z is a complex number. Therefore, in this paper, a solution to the pure Yang-Mills field equation comprises a classical field and quantum fluctuations. The classical field is a base of the quantum fluctuations, and corresponds physically to a vacuum. Components of the classical field transform under transformations of such as the SU(N) group including the SU(2) subgroup, whose finite fields are embedded in an extended group with classical zero fields. An example of such an extended group is SU(N) of the compact and simple group. By contrast, quantum fluctuations around the classical field are formulated by using the path integral of the fields with a symmetry represented by the extended group. No restriction is imposed on the quantum path integral, unlike the classical field case.

The solution to the differential equation of the Yang-Mills equation is restricted to the form given by Eq. (3.1). For this type of solution, the non-linear field strength tensor in Eq. (3.3) becomes linear. This linear solution is not a U(1) field but an SU(2) field, because the component of the field is multiplied by the generator T^a of the group. It is known that the potential between a particle and an antiparticle is derived from the Wilson loop in Eq. (3.46). In order to derive a linear potential in Eq. (3.67), a classical localized function $A_{(\mathrm{CL})}(x)$ is expressed in the form given by Eq. (3.1). This classical localized field does not satisfy the Yang-Mills equation. By adding an unlocalized field $A_{(\mathrm{CU})}(x)$ in Eq. (3.29) to the localized field $A_{(\mathrm{CL})}$ (with the charge density distribution such as that in Eq. (3.33)), the total field $A_{(\mathrm{CL})}(x)+A_{(\mathrm{CU})}(x)$ satisfies the Yang-Mills field equation given by Eq. (3.7). This is because the charges of these fields cancel each other. As described in Lemma 3.11, the added unlocalized field $A_{(\mathrm{CU})}(x)$ differs from the localized field, and does not influence the Wilson loop, as shown in Eq. (3.50). In the next procedure, the string tension (energy per unit length) for the linear potential is related to the size of the particle-antiparticle gluon pair, as indicated in Eq. (3.77). The decrease/increase in the energy of the pair is equal to the increase/decrease in the energy of the Yang-Mills field, as given in Eq. (3.75). The energy of the Yang-Mills field is derived from the energy-momentum tensor of the field in Eq. (3.69). In contrast to the atomic Bohr radius, which is obtained from fundamental physical units, the size of the pair of the particles is not determined, because this size is a given fundamental physical quantity. The binding energy at finite temperatures is derived from the Polyakov line in Eq. (3.79), and yield Eq. (3.82). This paper also shows that the Lagrangian density in the action functional has a local mass due to the included stationary field, as stated in Lemma 3.24. Finally, since quantum fluctuations for a sufficiently large cutoff energy lead to the asymptotic freedom with a small coupling constant, the potential derived from the Wilson loop reveals a Coulomb potential, as shown in Subsubsec 3.3.2.

Consequently, a solution to the mass gap problem is presented in Theorem 3.28.

The contents of this Sec. 3 are as follows.

[3.2] Confinement for Yang-Mills particles of gluons
[3.2.1] A classical solution of pure Yang-Mills fields
[3.2.2] Classical Wilson loop
[3.2.3] Scale-invariant energy of Yang-Mills fields, string tension of the linear potential, and Polyakov's binding energy
[3.3] Quantum field in path integral around the classical field as a vacuum
[3.3.1] Quantum action of pure Yang-Mills fields expanded in terms of scalar plane wave functions
[3.3.2] Quantum Wilson loop

3.2 Confinement for Yang-Mills particles of gluons

3.2.1 A classical solution of pure Yang-Mills fields

In the following, it is shown that the field expressed in Eq. (3.1) satisfies Eq. (3.8) without losing the properties of the SU(N) group, because the field is multiplied by the generator of the group.

Definition 3.1. For $x \in \mathbb{R}^4_{\mathrm{E}}$, let $A_{(\mathrm{C})\mu}(x)$ be a solution to the field equation of the classical pure Yang-Mills field. Here, using Definition 2.12, the present solution considered is that of SU(2) group embedded in SU(N) group in the form

$$A_{(\mathrm{C})\mu}(x) := \begin{cases} A^a_{(\mathrm{C})\mu}(x)T^a = \lambda^a \tilde{A}_{(\mathrm{C})\mu}(x)T^a & \text{for } a = 1,2,3 \\ \\ 0 & \text{for } a > 3 \end{cases}, \tag{3.1}$$

where λ^a are real-number constants.

Lemma 3.2. *For the classical Yang-Mills field in Eq. (3.1) with the structure constant f_{abc} in Definition 2.12, let's consider the following field strength*

$$F^a_{(\mathrm{C})\mu\nu} := \partial_\mu A^a_{(\mathrm{C})\nu} - \partial_\nu A^a_{(\mathrm{C})\mu} - g_{\mathrm{c}} \sum_{b,c} f^{abc} A^b_{(\mathrm{C})\mu} A^c_{(\mathrm{C})\nu}, \tag{3.2}$$

where g_{c} is a classical coupling constant. Then it follows that

$$F^a_{(\mathrm{C})\mu\nu} = \partial_\mu A^a_{(\mathrm{C})\nu} - \partial_\nu A^a_{(\mathrm{C})\mu}. \tag{3.3}$$

Proof. Since algebraic structure constants of the SU(N) group are totally antisymmetric denoted as $f_{abc} = -f_{acb}$, the following self-interaction term in Eq. (3.2) vanishes, to give

$$g_{\mathrm{c}} \sum_{b,c} f^{abc} A^b_{(\mathrm{C})\mu} A^c_{(\mathrm{C})\nu} = g_{\mathrm{c}} \tilde{A}_{(\mathrm{C})\mu} \tilde{A}_{(\mathrm{C})\nu} \sum_{b,c} \lambda^b \lambda^c f^{abc} = 0. \tag{3.4}$$

□

Before Lemma 3.3 below, it is shown that the variation in the Feynman gauge leads to the same result as that in Lemma 3.3. In the Feynman gauge, the action functional, which comprises the Lagrangian density in Eq. (2.22) and an additional term, is expressed as follows:

$$S_{(\mathrm{F})} := \int \, dx^4 \sum_a \left\{ -\frac{1}{4} [F^a_{\mu\nu}(x)]^2 - \frac{1}{2} [\partial_\mu A^a_\mu(x)]^2 \right\}. \tag{3.5}$$

Historically, the variational calculus for an action in analytical dynamics was constructed from differential equations. It is known that using Eq. (3.3), the above Eq. (3.5) for $A^a_{(\mathrm{C})\mu}$ is reduced to

$$\begin{aligned} S_{(\mathrm{FC})} \quad := \quad & -\int \, dx^4 \sum_a \frac{1}{2} \partial_\nu A^a_{(\mathrm{C})\mu} \partial_\nu A^a_{(\mathrm{C})\mu} \\ & -\int \, dx^4 \sum_a \frac{1}{2} \partial_\nu (A^a_{(\mathrm{C})\nu} \partial_\mu A^a_{(\mathrm{C})\mu} - A^a_{(\mathrm{C})\mu} \partial_\mu A^a_{(\mathrm{C})\nu}) \\ = \quad & -\int \, dx^4 \sum_a \frac{1}{2} \partial_\nu A^a_{(\mathrm{C})\mu} \partial_\nu A^a_{(\mathrm{C})\mu}. \end{aligned} \tag{3.6}$$

For the variation of $S_{(\mathrm{FC})}$ with respect to $A^a_{(\mathrm{C})\mu}(x)$, the stationary condition $\delta S_{(\mathrm{FC})} = 0$ also leads to the Yang-Mills field equation given by Eq. (3.8).

Lemma 3.3. *Under the condition $\partial_\mu A^a_{(\mathrm{C})\mu}$=0, the field equation $\partial_\mu F^a_{(\mathrm{C})\mu\nu} = 0$ for the classical field in Lemma 3.2 is reduced to*

$$\partial_\mu F^a_{(\mathrm{C})\mu\nu} = \partial_\mu \partial_\mu A^a_{(\mathrm{C})\nu} = 0. \tag{3.7}$$

Proof. Using the condition $\partial_\mu A^a_{(\mathrm{C})\mu}$=0, Eq. (3.3) leads to

$$\begin{aligned} \partial_\mu F^a_{(\mathrm{C})\mu\nu} &= \partial_\mu \partial_\mu A^a_{(\mathrm{C})\nu} - \partial_\nu \partial_\mu A^a_{(\mathrm{C})\mu} \\ &= \partial_\mu \partial_\mu A^a_{(\mathrm{C})\nu} = 0. \end{aligned} \tag{3.8}$$

□

It is noted that algebraic properties of non-Abelian Yang-Mills fields without self-interactions are different from those of Abelian fields, because the component of the Yang-Mills field $A^a(x)$ is multiplied by the generator matrix of such as the SU(N) group to form $A^a(x)T^a$, unlike Abelian fields such as electromagnetic fields.

Subsequently, the author presents a solution $A^a_{(\mathrm{C})\mu}(x)$ to the above differential equation in Eq. (3.8). A field configuration $A^a_{(\mathrm{CL})\mu}(x)$ in Eqs. (3.10)−(3.13) is considered, and the field yields a linear potential between a particle and an antiparticle, because $A^a_{(\mathrm{CL})\mu}(x)$ is multiplied by the group generator T^a. This $A^a_{(\mathrm{CL})\mu}(x)$ does not satisfy the differential equation in Eq. (3.8). By adding $A^a_{(\mathrm{CU})\mu}(x)$ in Eq. (3.29) to $A^a_{(\mathrm{CL})\mu}(x)$, a solution $A^a_{(\mathrm{C})\mu}(x)$ to Eq. (3.8) is derived.

Definition 3.4. The classical field configuration treated in this paper comprises two different functions

$$A^a_{(\mathrm{C})\mu}(x) = A^a_{(\mathrm{CL})\mu}(x) + A^a_{(\mathrm{CU})\mu}(x), \tag{3.9}$$

where $A^a_{(\mathrm{CL})\mu}$ and $A^a_{(\mathrm{CU})\mu}$ are localized and unlocalized fields, respectively.

It is noted that the classical Yang-Mills field is a base field of the quantum fields mentioned below. Physically, the base field is a vacuum, which may be a candidate for a black matter.

The above classical localized function is given in the restricted region. Physically, this is due to the property of a wave packet in the expanding universe with boundaries and the cutoff at the Planck energy.

Definition 3.5. For $t,x,y,z \in \mathbb{R}^4_{\mathrm{E}}$, the classical localized function is defined in the region $0 < T_{(\mathrm{b})0} \leq t \leq T_{(\mathrm{b})}$, $|x| \leq X_{(\mathrm{b})}$, $|y| \leq Y_{(\mathrm{b})}$, and $|z| \leq Z_{(\mathrm{b})}$, where the symbols used for the bounds are positive real numbers.

Definition 3.6. In the center-of-mass frame, which is physically used, the localized field $A^a_{(\mathrm{CL})\mu}$ for $1 \leq a \leq 3$ of the group SU(2) is embedded in the field of the group SU(N), and is shifted from $A^a_{(\mathrm{CL})\mu} = 0$ to the field with non-zero components. The classical localized field $A^a_{(\mathrm{CL})\mu}(x)$ are expressed by

$$
\begin{aligned}
A^a_{(\mathrm{CL})t}(x) &:= \lambda^a P_{(0)} s^{(0)}_{\mathrm{n}} \exp(-|c^{(0)}_\nu x_\nu|), && (3.10)\\
A^a_{(\mathrm{CL})x}(x) &:= 0, && (3.11)\\
A^a_{(\mathrm{CL})y}(x) &:= \lambda^a P_{(2)} s^{(2)}_{\mathrm{n}} \exp(-|c^{(2)}_\nu x_\nu|), && (3.12)\\
A^a_{(\mathrm{CL})z}(x) &:= \lambda^a P_{(3)} s^{(3)}_{\mathrm{n}} \exp(-|c^{(3)}_\nu x_\nu|), && (3.13)
\end{aligned}
$$

with

$$
P_{(0)} := \left(-\frac{1}{2} k_{N_{\mathrm{D}}}^{-1/2} b^{(0)}_\nu x_\nu\right) \frac{\exp(-|b^{(\mathrm{M})}_{\mu\nu} x_\mu x_\nu|)}{[1-\exp(-2|b^{(\mathrm{M})}_{\mu\nu} x_\mu x_\nu|)]^{1/2}}. \tag{3.14}
$$

For $a \geq 4$, the components are set as $A^a_{(\mathrm{CL})\mu} = 0$, and λ^a here are defined by constants that depend on the index a for the component of the symmetry group. In the center-of-mass frame, the symbols in Eqs. (3.10) and (3.12)−(3.14) are represented as $b^{(0)}_\nu$=$(0, a_{\mathrm{c}}, 0, 0)$ and $c^{(0)}_\nu = c^{(2)}_\nu = c^{(3)}_\nu = (0, 0, 1/d, 1/d)$, where $a_{\mathrm{c}} > 0$ and $d > 0$. The matrix $b^{(\mathrm{M})}_{\mu\nu}$ is written as

$$
b^{(\mathrm{M})}_{\mu\nu} := \begin{bmatrix} 0 & \frac{1}{2}a_{\mathrm{c}} & 0 & 0 \\ \frac{1}{2}a_{\mathrm{c}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.15}
$$

The variable $s^{(0)}_{\mathrm{n}}$ is a product of the form

$$
s^{(0)}_{\mathrm{n}} := s^{(0)}_{\mathrm{n}t} s^{(0)}_{\mathrm{n}x} s^{(0)}_{\mathrm{n}y} s^{(0)}_{\mathrm{n}z}, \tag{3.16}
$$

and $s^{(2)}_{\mathrm{n}}$ and $s^{(3)}_{\mathrm{n}}$ have the corresponding form, where

$$
\begin{aligned}
s^{(0)}_{\mathrm{n}t} &:= \quad s^{(2)}_{\mathrm{n}t} := \quad s^{(3)}_{\mathrm{n}t} := +1 && \text{for all } t,\\
s^{(0)}_{\mathrm{n}x} &:= \quad s^{(2)}_{\mathrm{n}x} := \quad s^{(3)}_{\mathrm{n}x} := +1 && \text{for all } x,\\
s^{(0)}_{\mathrm{n}y} &:= \quad s^{(2)}_{\mathrm{n}y} := \quad s^{(3)}_{\mathrm{n}y} := +1 && \text{for } y \geq 0,\\
s^{(0)}_{\mathrm{n}y} &:= -s^{(2)}_{\mathrm{n}y} := \quad s^{(3)}_{\mathrm{n}y} := -1 && \text{for } y < 0,\\
s^{(0)}_{\mathrm{n}z} &:= \quad s^{(2)}_{\mathrm{n}z} := \quad s^{(3)}_{\mathrm{n}z} := +1 && \text{for } z \geq 0,\\
s^{(0)}_{\mathrm{n}z} &:= \quad s^{(2)}_{\mathrm{n}z} := -s^{(3)}_{\mathrm{n}z} := -1 && \text{for } z < 0.
\end{aligned} \tag{3.17}
$$

The field $A^a_{(\mathrm{CL})t}$ for $x \geq 0$ is then represented by

$$
A^a_{(\mathrm{CL})t}(t, \mathbf{x}) = \lambda^a P_{(0)}(t, x) w_t(y) w_t(z), \tag{3.18}
$$

where

$$
P_{(0)}(t, x) = \frac{1}{2} k_{N_{\mathrm{D}}}^{-1/2} h(t, x), \tag{3.19}
$$

with

$$h(t,x) \quad := \quad \frac{-a_{\rm c}x\exp(-a_{\rm c}tx)}{[1-\exp(-2a_{\rm c}tx)]^{1/2}} \quad \text{for } x \geq \varepsilon_x, \tag{3.20}$$

$$h(t,x) \quad := \quad -h(t,-x) \qquad \text{for } x < -\varepsilon_x. \tag{3.21}$$

The symbol ε_x stands for an infinitesimal positive real number, and after the calculations a limit is taken as $\varepsilon_x \to 0$. The derivative defined is $\partial_x h = \lim_{\Delta x \to +0}(\Delta h/(\Delta x)$ for $x \geq \varepsilon_x$; and $\partial_x h = \lim_{\Delta x \to -0}(\Delta h/(\Delta x)$ for $x < -\varepsilon_x$. The function $w_t(y)$ is written as

$$w_t(y) := w(y) := \begin{cases} +\exp(-\frac{y}{d}) & \text{for } y \geq \varepsilon_y \\ \\ -\exp(+\frac{y}{d}) & \text{for } y < -\varepsilon_y \end{cases}, \tag{3.22}$$

where ε_y is an infinitesimal positive real number, and a limit is taken as $\varepsilon_y \to 0$. The derivative is defined by $\partial_y w = \lim_{\Delta y \to +0}(\Delta w/(\Delta y)$ for $y \geq \varepsilon_y$; and $\partial_y w = \lim_{\Delta y \to -0}(\Delta w/(\Delta y)$ for $y < -\varepsilon_y$. The forms for z are the same as those for y.

The fields $A^a_{({\rm CL})y}(t,\mathbf{x})$ and $A^a_{({\rm CL})z}(t,\mathbf{x})$ are written as

$$A^a_{({\rm CL})y}(t,\mathbf{x}) \quad = \quad \lambda^a \left(\frac{d}{2}\right)(\partial_t P_{(0)})w_y(y)w(z), \tag{3.23}$$

$$A^a_{({\rm CL})z}(t,\mathbf{x}) \quad = \quad \lambda^a \left(\frac{d}{2}\right)(\partial_t P_{(0)})w(y)w_z(z), \tag{3.24}$$

where

$$w_y(y) := \begin{cases} +w(y) & \text{for } y \geq \varepsilon_y \\ \\ -w(y) & \text{for } y < -\varepsilon_y \end{cases}, \tag{3.25}$$

$$w_z(z) := \begin{cases} +w(z) & \text{for } z \geq \varepsilon_z \\ \\ -w(z) & \text{for } z < -\varepsilon_z \end{cases}. \tag{3.26}$$

Lemma 3.7. *Let $A^a_{({\rm CL})\mu}(x)$ be the classical localized functions of the Yang-Mills field. Then $A^a_{({\rm CL})\mu}(x)$ satisfies the following Lorentz condition*

$$\partial_\mu A^a_{({\rm CL})\mu}(x) = 0, \tag{3.27}$$

where the derivatives are defined in the region in which the function is defined.

Proof. With the help of Definition 3.6 involving Eqs. (3.11), (3.18), and (3.22)−(3.26), it follows that

$$\begin{aligned} \partial_\mu A^a_{({\rm CL})\mu}(x) \quad = \quad & \lambda^a[(\partial_t P_{(0)})w(y)w(z) + 0 \\ & -\frac{1}{2}(\partial_t P_{(0)})w(y)w(z) - \frac{1}{2}(\partial_t P_{(0)})w(y)w(z)] = 0. \end{aligned} \tag{3.28}$$

□

The localized function $A^a_{(\mathrm{CL})\nu}(t,\mathbf{x})$ satisfies Eq. (3.32), where $\rho^a_\nu(t,\mathbf{x})$ is given by such as Eq. (3.33). The unlocalized function $A^a_{(\mathrm{CU})\nu}(t,\mathbf{x})$, which is different from $A^a_{(\mathrm{CL})\nu}(t,\mathbf{x})$, as shown in Eq. (3.39), is introduced in Eq. (3.29) to satisfy Eq. (3.36). Then $A^a_{(\mathrm{CL})\nu}(t,\mathbf{x})+A^a_{(\mathrm{CU})\nu}(t,\mathbf{x})$ becomes a solution to Eq. (3.38).

Definition 3.8. For $x \in \mathbb{R}^4_{\mathrm{E}}$, the unlocalized function in Eq. (3.9) has the form

$$\begin{aligned} A^a_{(\mathrm{CU})\mu}(t,\mathbf{x}) \quad := \quad & \int_{T_{(\mathrm{b})}}^{t} dt_s \int_{-X_{(\mathrm{b})}}^{X_{(\mathrm{b})}} dx_s \int_{-Y_{(\mathrm{b})}}^{Y_{(\mathrm{b})}} dy_s \int_{-Z_{(\mathrm{b})}}^{Z_{(\mathrm{b})}} dz_s \\ & \times G_4(t,\mathbf{x};t_s,\mathbf{x}_s)\rho^a_{(\mathrm{CU})\mu}(t_s,\mathbf{x}_s), \end{aligned} \tag{3.29}$$

where G_4 is the Green function denoted by

$$G_4(t,\mathbf{x};t_s,\mathbf{x}_s) := \frac{1}{2\pi^2}\frac{1}{(t-t_s)^2+(x-x_s)^2+(y-y_s)^2+(z-z_s)^2}, \tag{3.30}$$

which satisfies

$$\partial^2_\mu A^a_{(\mathrm{CU})\nu}(t,\mathbf{x}) = -\rho^a_{(\mathrm{CU})\nu}(t,\mathbf{x}), \tag{3.31}$$

with $\rho_{(\mathrm{CU})}$ being four-dimensional (4D) charge-current density of the classical unlocalized field. In the denominator in Eq. (3.30), an infinitesimal positive real number $\varepsilon_{\mathrm{Gr}}$ is added to $(t-t_s)^2+(x-x_s)^2+(y-y_s)^2+(z-z_s)^2$, and after the calculations the limit is taken as $\varepsilon_{\mathrm{Gr}} \to 0$. If necessary, the Green function is extended.

Lemma 3.9. *The classical localized function $A^a_{(\mathrm{CL})\nu}(t,\mathbf{x})$ satisfies*

$$\partial^2_\mu A^a_{(\mathrm{CL})\nu}(t,\mathbf{x}) = -\rho^a_{(\mathrm{CL})\nu}(t,\mathbf{x}) = -\rho^a_\nu(t,\mathbf{x}), \tag{3.32}$$

where $\rho^a_{(\mathrm{CL})\nu}(t,\mathbf{x}) = \rho^a_\nu(t,\mathbf{x})$ is the 4D charge-current density of the classical localized function $A^a_{(\mathrm{CL})\mu}(x)$ in Eq. (3.9), and is derived such as

$$\rho^a_t(t,x,y,z) = \begin{cases} \lambda^a\left[\left(\frac{1}{2}k_{N_{\mathrm{D}}}^{-1/2}\right)Q(t,x) - \frac{2}{d^2}P_{(0)}(t,x)\right]w(y)w(z) & \\ & \textit{for } \varepsilon_x \le x \le X_c \\ 0 & \textit{for } x > X_c \end{cases}, \tag{3.33}$$

where X_c is the wave-packet size of the classical localized function, and

$$Q(t,x) := a_{\mathrm{c}}^3x^3Q_1(t,x) - 2a_{\mathrm{c}}^2tQ_2(t,x) + a_{\mathrm{c}}^3t^2xQ_1(t,x), \tag{3.34}$$

with

$$\begin{aligned} Q_1(t,x) &:= Q_{1/2}(t,x)+4Q_{3/2}(t,x)+3Q_{5/2}(t,x), \\ Q_2(t,x) &:= Q_{1/2}(t,x)+Q_{3/2}(t,x), \\ Q_{1/2}(t,x) &:= \frac{\exp(-a_{\mathrm{c}}|tx|)}{[1-\exp(-2a_{\mathrm{c}}|tx|)]^{1/2}}, \\ Q_{3/2}(t,x) &:= \frac{\exp(-3a_{\mathrm{c}}|tx|)}{[1-\exp(-2a_{\mathrm{c}}|tx|)]^{3/2}}, \\ Q_{5/2}(t,x) &:= \frac{\exp(-5a_{\mathrm{c}}|tx|)}{[1-\exp(-2a_{\mathrm{c}}|tx|)]^{5/2}}. \end{aligned} \tag{3.35}$$

Proof. This lemma is obtained straightforward from Definition 3.6. $\square$

Lemma 3.10. *By setting*

$$\rho^a_{(\mathrm{CU})\nu}(t,\mathbf{x}) = -\rho^a_\nu(t,\mathbf{x}), \tag{3.36}$$

for the current density of the unlocalized field, the classical Yang-Mills field $A^a_{(\mathrm{C})\nu}(x)$ in Eq. (3.9) satisfies

$$\partial^2_\mu A^a_{(\mathrm{C})\nu}(x) = 0. \tag{3.37}$$

Proof. Using Eqs. (3.31) and (3.32),

$$\begin{aligned}
&\partial^2_\mu (A^a_{(\mathrm{CL})\nu}(x) + A^a_{(\mathrm{CU})\nu}(x)) \\
&= -\rho^a_{(\mathrm{CL})\nu}(t,\mathbf{x}) - \rho^a_{(\mathrm{CU})\nu}(t,\mathbf{x}) \\
&= -\rho^a_\nu(t,\mathbf{x}) + \rho^a_\nu(t,\mathbf{x}) = 0.
\end{aligned} \tag{3.38}$$

$\square$

Lemma 3.11. *The unlocalized field $A^a_{(\mathrm{CU})\nu}(t,\mathbf{x})$ differs from the localized field $A^a_{(\mathrm{CL})\nu}(t,\mathbf{x})$, namely,*

$$A^a_{(\mathrm{CU})\nu}(t,\mathbf{x}) \neq A^a_{(\mathrm{CL})\nu}(t,\mathbf{x}), \tag{3.39}$$

which states that Eq. (3.37) is meaningful as the Yang-Mills field.

Proof. From Eqs. (3.22) and (3.33), it follows that

$$\rho^a_t(t_s,x_s,-y_s,z_s) = -\rho^a_t(t_s,x_s,y_s,z_s), \tag{3.40}$$

$$\rho^a_t(t_s,x_s,y_s,-z_s) = -\rho^a_t(t_s,x_s,y_s,z_s), \tag{3.41}$$

canceling the charge contributions to the integral for the Green function in Eq. (3.29) at $y = 0$ and $z = 0$, and then the unlocalized field takes the value

$$A^a_{(\mathrm{CU})t}(t,\mathbf{x}) = 0 \quad \text{at } y = 0 \text{ and } z = 0. \tag{3.42}$$

By contrast, the localized function $A^a_{(\mathrm{CL})t}(t,\mathbf{x})$ does not always take the value of zero at $y = 0$ and $z = 0$, as can be seen, with the help of Eqs. (3.18)–(3.22). $\square$

3.2.2 Classical Wilson loop

In this subsection, a potential between a particle and an antiparticle for the stationary classical field is considered using the Wilson loop. These paired particles exist in the pure Yang-Mills field, and have opposite color charges, which imply the charges of the SU(N) symmetry. The Wilson loop is defined in Eq. (3.46) as trace of an exponential function of a closed line integral of the field. The trace obtains a physical quantity, and the potential in Eq. (3.67) is derived from the logarithmic (inverse) function of the exponential function.

Notably, properties of the group are used in such as Eq. (3.65), because the field component is multiplied by the generator of the group. Before presenting the Wilson loop for the Yang-Mills field, an example of the loop integral in the case of an Abelian field is briefly described. For $x \in \mathbb{R}^4_{\mathrm{E}}$, let's consider the following integral

$$\begin{aligned} I_{(\mathrm{CA})} &:= \oint dx_\mu \ A_\mu(x) = \oint dx_\mu \ A_\mu(t,x,y,z) \\ &= I^{(1)}_{(\mathrm{CA})} + I^{(2)}_{(\mathrm{CA})} + I^{(3)}_{(\mathrm{CA})} + I^{(4)}_{(\mathrm{CA})}, \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} I^{(1)}_{(\mathrm{CA})} &:= \int_{x_1}^{x_2} dx \ gA_x(t_1,x,0,0), \\ I^{(2)}_{(\mathrm{CA})} &:= \int_{t_1}^{t_2} dt \ gA_t(t,x_2,0,0), \\ I^{(3)}_{(\mathrm{CA})} &:= \int_{x_2}^{x_1} dt \ gA_x(t_2,x,0,0), \\ I^{(4)}_{(\mathrm{CA})} &:= \int_{t_2}^{t_1} dt \ gA_t(t,x_1,0,0). \end{aligned} \tag{3.44}$$

Here, a unit charge and a unit anticharge are in interaction with the field in the above $I^{(2)}_{(\mathrm{CA})}$ and $I^{(4)}_{(\mathrm{CA})}$, respectively, for a coupling constant g. By setting $A_x = 0$ in the nearly static case, it follows that

$$I_{(\mathrm{CA})} = g[A_t(t_1,x_2,0,0) - A_t(t_1,x_1,0,0)](t_2 - t_1), \tag{3.45}$$

which implies the potential depends on the distance between the charges. This loop integral is generalized to $\mathrm{Tr}[\exp(-iI_{(\mathrm{CA})})]$, which is available to both Abelian and non-Abelian fields.

Definition 3.12. The Wilson loop of the classical pure Yang-Mills field $A_{(\mathrm{C})\mu}(x)$ is defined by the form

$$W_{\mathrm{C}} := \mathrm{Tr}[\exp(-ig \oint dx_\mu \ A_{(\mathrm{C})\mu}(x))]. \tag{3.46}$$

To obtain the Wilson loop, the following line integrals are performed along a closed rectangular loop whose sides are parallel to the x- or t-axis

$$\begin{aligned} I^{a(1)}_{(\mathrm{CL})} &:= \int_{x_1}^{x_2} dx \ A^a_{(\mathrm{CL})x}(t_1,x,0,0), \\ I^{a(2)}_{(\mathrm{CL})} &:= \int_{t_1}^{t_2} dt \ A^a_{(\mathrm{CL})t}(t,x_2,0,0), \\ I^{a(3)}_{(\mathrm{CL})} &:= \int_{x_2}^{x_1} dx \ A^a_{(\mathrm{CL})x}(t_2,x,0,0), \\ I^{a(4)}_{(\mathrm{CL})} &:= \int_{t_2}^{t_1} dt \ A^a_{(\mathrm{CL})t}(t,x_1,0,0), \end{aligned} \tag{3.47}$$

where $x_2 = -x_1$.

Lemma 3.13. *The line integrals contained in the Wilson loop for the classical unlocalized field vanish.*

Proof. The x-component of the unlocalized field satisfies $A^a_{(\mathrm{CU})x}(t,\mathbf{x})=0$ from Eqs. (3.11), (3.29), (3.32), and (3.36), then the line integrals along the x-axis vanish as written by

$$\begin{aligned} I^{a(1)}_{(\mathrm{CU})} &:= \int_{x_1}^{x_2} dx\ A^a_{(\mathrm{CU})x}(t_1,x,0,0)=0,\\ I^{a(3)}_{(\mathrm{CU})} &:= \int_{x_2}^{x_1} dx\ A^a_{(\mathrm{CU})x}(t_2,x,0,0)=0. \end{aligned} \tag{3.48}$$

In addition, with the aid of Eq. (3.42), the following relations are derived

$$\begin{aligned} I^{a(2)}_{(\mathrm{CU})} &:= \int_{t_1}^{t_2} dx\ A^a_{(\mathrm{CU})t}(t,x_1,0,0)=0,\\ I^{a(4)}_{(\mathrm{CU})} &:= \int_{x_1}^{x_2} dx\ A^a_{(\mathrm{CU})t}(t,x_2,0,0)=0. \end{aligned} \tag{3.49}$$

From Eqs. (3.48) and (3.49), the sum of the contributions of the classical unlocalized field to the Wilson loop thus vanishes as

$$I^a_{(\mathrm{CU})} := I^{a(1)}_{(\mathrm{CU})}+I^{a(2)}_{(\mathrm{CU})}+I^{a(3)}_{(\mathrm{CU})}+I^{a(4)}_{(\mathrm{CU})}=0. \tag{3.50}$$

□

Lemma 3.14. *The total line integral $I^a_{(\mathrm{CL})}$ in the Wilson loop for the classical localized field has the form*

$$I^a_{(\mathrm{CL})} = -\lambda^a (k_{N_\mathrm{D}}^{-1/2}) \arccos[\exp(-a_\mathrm{c}x_2t_2)]. \tag{3.51}$$

Proof. From Eq. (3.11), the line integrals along the x-axis in Eq. (3.47) yield

$$I^{a(1)}_{(\mathrm{CL})}=0, \qquad I^{a(3)}_{(\mathrm{CL})}=0. \tag{3.52}$$

Whereas, using Definition 3.6 that contains Eqs. (3.18)$-$(3.22), the integrals along the t-axis in Eq. (3.47) provide

$$I^{a(2)}_{(\mathrm{CL})}+I^{a(4)}_{(\mathrm{CL})} = \frac{\lambda^a k_{N_\mathrm{D}}^{-1/2}}{2}[H_{(\mathrm{CL})}(t_2,x_2)-H_{(\mathrm{CL})}(t_1,x_2)], \tag{3.53}$$

where $x_2=-x_1$. From Eqs. (3.20) and (3.21), it follows that

$$H_{(\mathrm{CL})}(t,x) := \int dt'\ h(t',x) = -\arccos[\exp(-a_\mathrm{c}xt)], \tag{3.54}$$

where $H(t,x)$ has the symmetry $H(t,x)=-H(t,-x)$, and summing the above integrals results in

$$\begin{aligned} I^a_{(\mathrm{CL})} &= I^{a(1)}_{(\mathrm{CL})}+I^{a(2)}_{(\mathrm{CL})}+I^{a(3)}_{(\mathrm{CL})}+I^{a(4)}_{(\mathrm{CL})}\\ &= -\lambda^a k_{N_\mathrm{D}}^{-1/2}\{\arccos[\exp(-a_\mathrm{c}x_2t_2)]\\ &\quad -\arccos[\exp(-a_\mathrm{c}x_2t_1)]\}. \end{aligned} \tag{3.55}$$

By neglecting the last term $\arccos[\exp(-a_\mathrm{c}x_2t_1)]$ for small $a_\mathrm{c}x_2t_1$, Eq. (3.51) is thus derived. □

Lemma 3.15. *Only the classical localized field contributes to the total line integral $I^a_{(\mathrm{C})}$ in the Wilson loop, as written by*

$$I^a_{(\mathrm{C})} = I^a_{(\mathrm{CL})}. \tag{3.56}$$

Proof. With the aid of Eqs. (3.50) and (3.55), the Wilson loop is reduced to the following integral to give the above lemma

$$I^a_{(\mathrm{C})} = I^a_{(\mathrm{CL})} + I^a_{(\mathrm{CU})} = I^a_{(\mathrm{CL})}. \tag{3.57}$$

□

Lemma 3.16. *The classical Wilson loop yields a classical potential V_{C} between a charge and an anticharge of the Yang-Mills field of the form*

$$V_{\mathrm{C}} = \frac{a_{\mathrm{c}}}{2}(x_2 - x_1). \tag{3.58}$$

Proof. Using Eq. (3.51), the classical Wilson loop in Eq. (3.46) becomes

$$W_{\mathrm{C}} = \mathrm{Tr}[\exp(i\sum_a y' g \lambda^a k_{N_{\mathrm{D}}}^{-1/2} T^a)], \tag{3.59}$$

with

$$y' := \arccos[\exp(-a_{\mathrm{c}} x_2 t_2)])]. \tag{3.60}$$

The value of the normalization constant $k_{N_{\mathrm{D}}}$ for $\lambda^1 = \lambda^2 = \lambda^3$ is given to satisfy

$$\begin{aligned} & k_{N_{\mathrm{D}}}^{-1} g_{\mathrm{c}}^2 \sum_{a=1}^{3}\sum_{b=1}^{3} \lambda^a \lambda^b T^a T^b \\ &= \sum_{a=1}^{3}\sum_{b=1}^{3} (k_{N_{\mathrm{D}}}^{-1/2} g_{\mathrm{c}} \lambda^a T^a)(k_{N_{\mathrm{D}}}^{-1/2} g_{\mathrm{c}} \lambda^b T^b) \\ &= I^{(3)}, \end{aligned} \tag{3.61}$$

where $I^{(3)}$ is the unit matrix in the form of 2×2 submatrices, such as SU(2), embedded in the $N \times N$ matrix, which has individual elements equal to zero. For the quantity in the above equation, ξ^a is defined by

$$\xi^a := k_{N_{\mathrm{D}}}^{-1/2} g_{\mathrm{c}} \lambda^a. \tag{3.62}$$

Then, the following series are considered

$$\begin{aligned} \cos(y) &= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}, \\ \sin(y) &= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}. \end{aligned} \tag{3.63}$$

With the help of the properties of the matrices T^a, representing generators of the SU(2) group,

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta_{ab}, \quad \mathrm{Tr}(T^a) = 0, \tag{3.64}$$

$\mathrm{Tr}[\exp(iy'\sum_{a=1}^{3}\xi^aT^a)]$ is simplified as

$$\begin{aligned}
&\mathrm{Tr}[\exp(iy'\sum_{a=1}^{3}\xi^aT^a)] \\
&= \mathrm{Tr}[\cos(y'\sum_{a=1}^{3}\xi^aT^a)] + i\mathrm{Tr}[\sin(y'\sum_{a=1}^{3}\xi^aT^a)] \\
&= \mathrm{Tr}[\cos(y'I^{(3)})] = [\mathrm{Tr}(I^{(3)})]\cos(y').
\end{aligned} \tag{3.65}$$

Then, using Eq. (3.59), the classical Wilson loop, which is obtained as a sum of the integrals in Eq. (3.47) for small $a_\mathrm{c}x_2t_1$, becomes

$$\begin{aligned}
W_\mathrm{C} &= [\mathrm{Tr}(I^{(3)})]\left\{\cos\{\arccos[\exp(-a_\mathrm{c}x_2t_2)]\}\right\} \\
&= [\mathrm{Tr}(I^{(3)})]\exp(-a_\mathrm{c}x_2t_2) \\
&= \exp[-a_\mathrm{c}x_2(t_2-t_1)+\ln[\mathrm{Tr}(I^{(3)})]] \\
&= \exp\left[-\frac{a_\mathrm{c}}{2}(x_2-x_1)(t_2-t_1)+\ln[\mathrm{Tr}(I^{(3)})]\right],
\end{aligned} \tag{3.66}$$

where the relation $x_2 = -x_1$ has been used. It then follows that

$$V_\mathrm{C} = -\frac{\ln(W_\mathrm{C})}{t_2-t_1} = \frac{a_\mathrm{c}}{2}(x_2-x_1), \tag{3.67}$$

where the last term in Eq. (3.66) was neglected. □

It is note that that this linear potential is not derived for Abelian fields, since the Abelian fields do not involve the matrices T^a of the group.

3.2.3 Scale-invariant energy of Yang-Mills fields, string tension of the linear potential, and Polyakov's binding energy

The Yang-Mills field contains a basic constant that is not determined theoretically. This constant is a given scale-invariant energy that indicates the size of a pair composed of a particle and an antiparticle. In the case of an atom, this size corresponds to an atomic size, which is determined using the fundamental constants such as the Planck constant and electronic charge. The liner string tension is related to the above scale-invariant energy. Furthermore, the binding energy between a particle and an antiparticle is derived for the pure Yang-Mills field. In these analysis processes, the energy of the Yang-Mills field is calculated using Eq. (3.69), and Eq. (3.74) is obtained. The decrease/increase in the energy of the Yang-Mills field is equal to the increase/decrease in the energy of the paired particles through the linear potential. This equality provides the relationship between the slope of the linear potential and the distance between the paired particles, as indicated in Eq. (3.77). The binding energy, which is the mass of the pair of the particles, is calculated using the Polyakov line in Eq. (3.79), yielding Eq. (3.81).

Physically, an atom has the characteristic Bohr radius, which is expressed in terms of Planck's constant, electron mass, and coupling constant in the Abelian field theory. By contrast, it is known that non-Abelian Yang-Mills fields have a somewhat different intrinsic scale-invariant energy, which is equivalently length or time. This fundamental scale-invariant energy is a given constant similar to such as the Planck's constant, and cannot be determined from theory. It is therefore required that a solution to the Yang-Mills equation be consistent with this scale-invariance. Here, the present theory introduces the scale-invariant energy of the classical field at as a vacuum Minkowski time t', corresponding to Euclidean time t.

Definition 3.17. Let $\varepsilon_{t'}$ be a positive real number in Minkowski time t'. Then the existence of classical fields is limited to $|x| \leq \varepsilon_{t'}$, where $c = 1$ with the speed of light c.

Lemma 3.18. *The energy of the classical localized field $E_{(\mathrm{CL})}(\varepsilon_{t'})$ at the Minkowski time t' is given by*

$$E_{(\mathrm{CL})}(\varepsilon_{t'}) = \frac{\zeta(3)\Gamma(3)}{4g_{\mathrm{c}}^2}\frac{1}{a_{\mathrm{c}}(\varepsilon_{t'})^3}, \tag{3.68}$$

where ζ and Γ are zeta and gamma functions, respectively.

Proof. The energy of the classical localized field $E_{(\mathrm{CL})}(\varepsilon_{t'})$ is expressed with the aid of the energy-momentum tensor [32] $T_{00(\mathrm{CL})}$

$$E_{(\mathrm{CL})} = \int dxdydz \;\; T_{00(\mathrm{CL})}. \tag{3.69}$$

Here

$$T_{00(\mathrm{CL})} = \frac{\partial L_{\mathrm{F(CL)}}}{\partial(\frac{\partial A_{(\mathrm{CL})\nu}}{\partial x'_0})}\frac{\partial A_{(\mathrm{CL})\nu}}{\partial x'_0} - L_{\mathrm{F(CL)}}, \tag{3.70}$$

where

$$L_{\mathrm{F(CL)}} := -\frac{2}{4}\mathrm{Tr}(F_{(\mathrm{CL})\mu\nu})^2. \tag{3.71}$$

Then let us use the aforementioned field strength $F_{(\mathrm{CL})\mu\nu}$ at time t' in the limit as $d \to 0$ with d being the thickness of the classical localized field. The dominant term is the integral of squared field strengths $[-\partial_y A^a_{(\mathrm{CL})t'}]^2$ in $[F^a_{(\mathrm{CL})yt'}]^2$ and $[-\partial_z A^a_{(\mathrm{CL})t'}]^2$ in $[F^a_{(\mathrm{CL})zt'}]^2$, leading to

$$\begin{aligned} E_{(\mathrm{CL})}(\varepsilon_{t'}) \;\; = \;\; & \{2\int_0^\infty dx \;\; [\frac{1}{g_{\mathrm{c}}}h(\varepsilon_{t'},x)]^2\} \\ & \times\frac{4}{2}\{\int_0^\infty\int_0^\infty dydz \;\; [(\partial_y w(y)w(z))^2 + (w(y)\partial_z w(z))^2]\}. \end{aligned} \tag{3.72}$$

This integral does not depend on d, and limits are taken as $\varepsilon_y \to 0$, $\varepsilon_z \to 0$, and $\varepsilon_x \to 0$. Using Eqs. (3.20)−(3.22) and

$$\int_0^\infty dx(h(\varepsilon_{t'},x))^2 = \zeta(3)\Gamma(3)a_{\mathrm{c}}^2\frac{1}{(2a_{\mathrm{c}}\varepsilon_{t'})^3}, \tag{3.73}$$

the integral in (3.72) results in

$$E_{(\mathrm{CL})}(\varepsilon_{t'}) = \frac{\zeta(3)\Gamma(3)}{4g_{\mathrm{c}}^2}\frac{1}{(\varepsilon_{t'})^3 a_{\mathrm{c}}}. \tag{3.74}$$

□

Lemma 3.19. *For charges of gluons, namely, particles of the Yang Mills field, let's consider a pair of a charge and an anticharge, which are separated by the distance $2\varepsilon_{t'}$ with $c=1$, where c is physically the speed of light. Let $E_{(\mathrm{CL})}(\varepsilon_{t'})$ be the energy of the classical localized field, and let $E_{(\mathrm{LP})}(\varepsilon_{t'}) = |-\frac{a_{\mathrm{c}}}{2}(2\varepsilon_{t'})| = |-\sigma(2\varepsilon_{t'})|$ be the stabilized/unstabilized energy of the pair. The following relation can be set*

$$\begin{aligned} E_{(\mathrm{CL})}(\varepsilon_{t'}) &= E_{(\mathrm{LP})}(\varepsilon_{t'}), & (3.75)\\ \frac{1}{\lambda_{\mathrm{MOM}}} &= 4\varepsilon_{t'}, & (3.76)\end{aligned}$$

with λ_{MOM} being the intrinsic scale-invariant energy. This λ_{MOM} is not determined from theory. Then, the absolute string tension (energy per unit length) σ has the form

$$\sigma = \frac{1}{2}a_{\mathrm{c}} = \frac{4\zeta(3)^{1/2}\Gamma(3)^{1/2}\lambda_{\mathrm{MOM}}^2}{g_{\mathrm{c}}}. \tag{3.77}$$

Proof. With the use of Eqs. (3.74)−(3.76), the above Eq.(3.77) is obtained. □

The binding energy of a charge-anticharge pair is obtained from the Polyakov line [33] in a similar way to the Wilson loop.

Definition 3.20. Corresponding to physics, let k_B and T be the Boltzmann constant and temperature, respectively. Then τ is defined by $\tau = 1/k_BT$. By replacing t by τ' and under the periodic condition along the temperature axis, the classical localized function at finite T has the following form

$$\begin{aligned} &A^a_{(\mathrm{CL})t}(t,x,y,z) \to A^a_{(\mathrm{CL})\tau}(\tau',x,y,z)\\ &= A^a_{(\mathrm{CL})t}(\tau',x,y,z) + A^a_{(\mathrm{CL})t}(\tau-\tau',x,y,z). \end{aligned} \tag{3.78}$$

The Polyakov line is defined by the line integral

$$P_\tau := \mathrm{Tr}[\exp(-ig_{\mathrm{c}}\sum_a I^a_{\mathrm{P}} T^a)], \tag{3.79}$$

with

$$I^a_{\mathrm{P}} := \int_{\tau_\varepsilon}^{\tau-\tau_\varepsilon} d\tau' \ A^a_{(\mathrm{CL})\tau}(\tau', x_2, 0, 0), \tag{3.80}$$

where the starting point returns to the end point, and τ_ε is an infinitesimally small real number. It is known that the Polyakov line yields the following binding energy of a charge-anticharge pair

$$\varepsilon_{\mathrm{q}} = -\ln(P_\tau). \tag{3.81}$$

Lemma 3.21. *For the charge-anticharge distance r, the Polyakov line gives the following binding energy*

$$\varepsilon_{\mathrm{q}} = \frac{\sigma r}{k_{\mathrm{B}} T}. \tag{3.82}$$

This equation states that for large τ, physically at low temperatures, the non-zero binding energy leads to a finite mass gap of the pure Yang-Mills field due to the pairing of a charge and an anticharge.

Proof. With the aid of Eqs. (3.55) and (3.66), the Polyakov line is represented by

$$\begin{aligned} P_\tau &= \cos\{\arccos[\exp(-\sigma r \tau)] \\ &\quad - \arccos[\exp(-\sigma r \tau_\varepsilon)]\}, \end{aligned} \tag{3.83}$$

where constant was neglected. When $\tau_\varepsilon \to 0$, this P_τ becomes

$$P_\tau = \cos\{\arccos[\exp(-\sigma r \tau)]\} = \exp(-\sigma r \tau). \tag{3.84}$$

Denoting E_{B} by $E_{\mathrm{B}} = \sigma r$, the binding energy is thus derived as

$$\begin{aligned} \varepsilon_{\mathrm{q}} &= -\ln(P_\tau) = -\ln\left[\exp\left(\frac{-E_{\mathrm{B}}}{k_{\mathrm{B}} T}\right)\right] \\ &= \frac{E_{\mathrm{B}}}{k_{\mathrm{B}} T} = \frac{\sigma r}{k_{\mathrm{B}} T}. \end{aligned} \tag{3.85}$$

□

3.3 Quantum field in path integral around the classical field as a vacuum

3.3.1 Quantum action of pure Yang-Mills fields expanded in terms of scalar plane wave functions

Here, the additional potential between a particle and an antiparticle is derived considering quantum fluctuations from the aforementioned stationary state. The Wilson loop for quantum fluctuations is calculated using the Feynman path integral. For the sufficiently large cutoff value, the coupling constant is small, and the action functional in the Feynman gauge is given by Eq. (3.88). By expanding the quantum field in terms of the plane-wave basis functions in Eq. (3.92), the path integral is reduced to the Gauss integral in Eq. (3.108). The potential is independent of the generators of the group in Eq. (3.112), and has the same form as that of the U(1) group. The potential reveals to be a Coulomb potential, and the absolute Coulomb potential is larger than the absolute linear potential at short distances. As aforementioned, the pure Yang-Mills field $A^a_\mu(x)$ in the present formalism is expressed as

$$A^a_\mu(x) = A^a_{(\mathrm{C})\mu}(x) + A^a_{(\mathrm{Q})\mu}(x), \tag{3.86}$$

where $A_{(\mathrm{C})\mu}(x)$ is the classical field as a vacuum and $A^a_{(\mathrm{Q})\mu}(x)$ is the quantum field, which is the fluctuation around the classical field. When necessary, Faddeev-Popov (FP) ghost is added. Then, the zeroth order of the action for the non-Abelian Yang-Mills fields is the classical field, and the first-order action vanishes due to the classical equations of motion, while the second order remains.

Definition 3.22. Let's consider the following quantum action with the coupling constant g in the Feynman gauge

$$S = S^{(2)} + S^{(3)} + S^{(4)}, \tag{3.87}$$

where

$$\begin{aligned}
S^{(2)} &:= -\frac{1}{2}\sum_a \int d^4x\ (\partial_\nu A^a_{(\mathrm{Q})\mu}\partial_\nu A^a_{(\mathrm{Q})\mu}), && (3.88)\\
S^{(3)} &:= g\sum_{a,b,c} f^{abc}\int d^4x\ (A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{Q})\nu}\partial_\mu A^a_{(\mathrm{C})\nu} \\
&\quad +A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{C})\nu}\partial_\mu A^a_{(\mathrm{Q})\nu}+A^b_{(\mathrm{C})\mu}A^c_{(\mathrm{Q})\nu}\partial_\mu A^a_{(\mathrm{Q})\nu} \\
&\quad +A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{Q})\nu}\partial_\mu A^a_{(\mathrm{Q})\nu}), && (3.89)\\
S^{(4)} &:= -\frac{g^2}{4}\sum_{a,b,c,d,e} f^{abc}f^{ade}\int d^4x\ (A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{Q})\nu}A^d_{(\mathrm{C})\mu}A^e_{(\mathrm{C})\nu} \\
&\quad +A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{C})\nu}A^d_{(\mathrm{Q})\mu}A^e_{(\mathrm{C})\nu}+A^b_{(\mathrm{C})\mu}A^c_{(\mathrm{Q})\nu}A^d_{(\mathrm{Q})\mu}A^e_{(\mathrm{C})\nu} \\
&\quad +A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{C})\nu}A^d_{(\mathrm{C})\mu}A^e_{(\mathrm{Q})\nu}+A^b_{(\mathrm{C})\mu}A^c_{(\mathrm{Q})\nu}A^d_{(\mathrm{C})\mu}A^e_{(\mathrm{Q})\nu} \\
&\quad +A^b_{(\mathrm{C})\mu}A^c_{(\mathrm{C})\nu}A^d_{(\mathrm{Q})\mu}A^e_{(\mathrm{Q})\nu}+A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{Q})\nu}A^d_{(\mathrm{Q})\mu}A^e_{(\mathrm{C})\nu} \\
&\quad +A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{Q})\nu}A^d_{(\mathrm{C})\mu}A^e_{(\mathrm{Q})\nu}+A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{C})\nu}A^d_{(\mathrm{Q})\mu}A^e_{(\mathrm{Q})\nu} \\
&\quad +A^b_{(\mathrm{C})\mu}A^c_{(\mathrm{Q})\nu}A^d_{(\mathrm{Q})\mu}A^e_{(\mathrm{Q})\nu}+A^b_{(\mathrm{Q})\mu}A^c_{(\mathrm{Q})\nu}A^d_{(\mathrm{Q})\mu}A^e_{(\mathrm{Q})\nu}). \\
& && (3.90)
\end{aligned}$$

The above quantum action comprises kinetic and self-interaction terms, and the latter in the cubic and quartic form almost contain the classical field. The cubic term is small because the quantum coupling constant g is small for short distances between spacetime elements due to the asymptotic freedom; and the quartic term is smaller than the cubic term for such short inter-element distances.

The contributions of the quantum fluctuations are analyzed using the path integral. The mass gap problem for the pure Yang-Mills field in the present case is for low-energy phenomena, and particles of the Yang-Mills field with the following form are not excited beyond the first cutoff energy.

Definition 3.23. Using Definition 2.13, the quantum pure Yang-Mills fields are expanded in terms of basis functions $\chi_k(x)$ within the first cutoff energy as

$$A^a_{(\mathrm{Q})\mu}(x) = \sum_k A^a_{(\mathrm{Q})\mu k}\chi_k(x), \tag{3.91}$$

where considering Definition 2.7, the basis functions $\chi_k(x)$ are denoted by

$$\chi_k(x) = \exp(ik_\mu x_\mu) \quad \text{with } |k| \le \pi/a_r. \tag{3.92}$$

Since the energy of the localized field is independent of d in Eq. (3.22), one can consider the case in which a limit is taken as $d \to 0$ for the classical field. Before the lemma below, let us consider a scalar field $\phi_\mathrm{M}(x)$

with a mass m_{M}, which was introduced by Klein and Gordon, and was used by Yukawa. The Lagrangian density of the mass term for this field has the form

$$\mathscr{L}_{\mathrm{M}} \propto \frac{1}{2}\phi_{\mathrm{M}}(x)(m_{\mathrm{M}})^2\phi_{\mathrm{M}}(x). \tag{3.93}$$

Lemma 3.24. *Let $S^{(41)}_{(\mathrm{CQ})}$ be an integral denoted by*

$$S^{(41)}_{(\mathrm{CQ})} := -\frac{1}{4}g^2 f^{abc} f^{ade} \int d^4x \ A^b_{(\mathrm{Q})\mu}(x) A^c_{(\mathrm{C})t}(x) A^e_{(\mathrm{C})t}(x) A^d_{(\mathrm{Q})\mu}(x). \tag{3.94}$$

For $b = d$ and $c = e$, it follows that

$$S^{(41)}_{(\mathrm{CQ})} = -\frac{1}{4}g^2 (f^{abc})^2 \int d^4x \ A^b_{(\mathrm{Q})\mu}(x)[A^c_{(\mathrm{C})t}(x)]^2 A^b_{(\mathrm{Q})\mu}(x). \tag{3.95}$$

As aforementioned, the summation notation is employed solely for the indices referring to spacetime coordinates. In the case of $A^c_{(\mathrm{C})t}(x) \neq 0$, from the above equation, the relation,

$$m_{\mathrm{loc}} = \frac{1}{4}g^2 (f^{abc})^2 [A^c_{(\mathrm{C})t}(x)]^2 > 0, \tag{3.96}$$

states that the pure Yang-Mill field has a local mass gap of $(2m_{\mathrm{loc}})^{1/2}$.

Proof. In Eq. (3.96), squared non-zero structure constants take the positive value as written by

$$(f^{abc})^2 > 0. \tag{3.97}$$

Hence, m_{loc} and $(2m_{\mathrm{loc}})^{1/2}$ have each positive value, with the aid of Eq. (3.93). □

3.3.2 Quantum Wilson loop

Here, the dominant quantum fluctuations to the Wilson loop is presented, considering that higher-order terms do no contribute to the loop, due to the asymptotic freedom having small coupling at short distances between spacetime elements via renormalizations.

Definition 3.25. Let's denote the action of the kinetic terms in Eq. (3.88) by $S^{(2)a}_{\mathrm{Q,Q}}$. Then this term is expanded in terms of the basis functions to give

$$\begin{aligned} S^{(2)a}_{\mathrm{Q,Q}} &:= -\frac{1}{2}\int d^4x \ [\partial_\nu \sum_k A^a_{(\mathrm{Q})\mu k}\chi_k(x)][\partial_\nu \sum_{k'} A^a_{(\mathrm{Q})\mu k'}\chi_{k'}(x)] \\ &= \sum_{k,k'} A^a_{(\mathrm{Q})\mu k} M^{\mathrm{Q,Q}}_{k'} A^a_{(\mathrm{Q})\mu k'}, \end{aligned} \tag{3.98}$$

where

$$M^{\mathrm{Q,Q}}_{kk'} := -\frac{1}{2}\int d^4x \ [\partial_\nu \chi_k(x)][\partial_\nu \chi_{k'}(x)], \tag{3.99}$$

with each diagonal eigenvalue $\eta_{(\mu)k}$ in the form proportional to k^2 for the plane-wave basis functions.

Definition 3.26. The quantum action S_{Q} considered for the above $S^{(2)a}_{\mathrm{Q,Q}}$ is expressed by

$$S_{\mathrm{Q}} = \sum_a \sum_{k,k'} M^{\mathrm{Q,Q}}_{kk'} A^a_{(\mathrm{Q})\mu k} A^a_{(\mathrm{Q})\mu k'}. \tag{3.100}$$

By using the classical Wilson loop W_{C}, in which the 2×2 unit submatrix $I^{(3)}$ embedded in the $N\times N$ matrix with vanishing components, the Wilson loop W_{Q} in the quantum path integral has the form

$$W_{\mathrm{Q}} = \mathrm{Tr}\left\{ W_{\mathrm{C}} I^{(3)} \frac{1}{Z_{\mathrm{N}}} \int D[A^a_{(\mathrm{Q})\mu k}] \;\; \exp(-S_{\mathrm{Q}}) \exp(C) \right\}. \tag{3.101}$$

Here, C is denoted by

$$C := -ig \sum_a \sum_k \beta_{\mu k} A^a_{(\mathrm{Q})\mu k} T^a, \tag{3.102}$$

where

$$\beta_{\mu k} := \oint dx_\mu \;\; \chi_k(x). \tag{3.103}$$

The quantity Z_{N} is a normalization constant in the path integral with respect to $A^a_{(\mathrm{Q})\mu k}$, and is written as

$$Z_{\mathrm{N}} := \int D[A^a_{(\mathrm{Q})\mu k}] \;\; \exp(-S_{\mathrm{Q}}). \tag{3.104}$$

In Eq. (3.101), the contribution from the classical field gives rise to the prefactor. Meanwhile, the quantum fluctuations $A^a_{(\mathrm{Q})\mu k}$ of the pure Yang-Mills field such as the SU(3) gauge field are represented in terms of the matrices T^a that represent the SU(3) group as an example.

Lemma 3.27. *The analytical confining potential $V(x_2 - x_1)$ between a Yang-Mills charge and an anticharge separated by the distance $x_2 - x_1$ is composed of the first classical linear term plus the second quantum Coulomb term written by*

$$V(x_2 - x_1) = \frac{a_{\mathrm{c}}}{2}(x_2 - x_1) + V_{\mathrm{C}}(x_2 - x_1). \tag{3.105}$$

Proof. By the diagonalization of Eq. (3.100) using a diagonal matrix $R_{kk'}$ associated with the eigenvalues $\eta_{(\mu)k}$ in the general case

$$S_{\mathrm{Q}} = \sum_a \sum_k \eta_{(\mu)k} (A'^a_{(\mathrm{Q})\mu k})^2, \tag{3.106}$$

with

$$A'^a_{(\mathrm{Q})\mu k} := R_{kk'} A^a_{(\mathrm{Q})\mu k'}, \tag{3.107}$$

it follows that

$$\begin{aligned} W_{\mathrm{Q}} \quad = \quad & \mathrm{Tr}\left\{ W_{\mathrm{C}} I^{(3)} \frac{1}{Z_{\mathrm{N}}} \int D[A'^a_{(\mathrm{Q})\mu k}] \;\; \exp(-i \sum_a \sum_k B'^a_{\mu k} A'^a_{(\mathrm{Q})\mu k}) \right. \\ & \left. \times \exp(-\sum_a \sum_k \eta_{(\mu)k} (A'^a_{(\mathrm{Q})\mu k})^2) \right\}, \end{aligned} \tag{3.108}$$

where

$$B'^{a}_{\mu k} := g\sum_{k'} \beta_{\mu k'} R^{-1}_{kk'} T^a. \tag{3.109}$$

The integrals have the form of Gaussian integrals, and the odd term vanishes independently of the group, leading to

$$\begin{aligned} W_{\mathrm{Q}} &= \mathrm{Tr}\Big\{ W_{\mathrm{C}} I^{(3)} \frac{1}{Z_{\mathrm{N}}} \Pi_a \Pi_k \int dA'^{a}_{(\mathrm{Q})\mu k} \\ &\quad \times \cos(B'^{a}_{\mu k} A'^{a}_{(\mathrm{Q})\mu k}) \exp[-\eta_{(\mu)k}(A'^{a}_{(\mathrm{Q})\mu k})^2] \Big\}. \end{aligned} \tag{3.110}$$

By denoting $B'^{a}_{\mu k}$ as $B'^{a}_{\mu k} = \tilde{B}'_{\mu k} T^a$, and with the aid of

$$Z_{\mathrm{N}} = \Pi_a \Pi_k \left(\frac{\pi}{\eta_{(\mu)k}} \right)^{1/2}, \tag{3.111}$$

the quantum Wilson loop results in

$$W_{\mathrm{Q}} = \mathrm{Tr}\left[W_{\mathrm{C}} I^{(3)} \exp\left(-\sum_a \sum_k \frac{(\tilde{B}'_{\mu k})^2}{4\eta_{(\mu)k}} T^a T^a \right) \right]. \tag{3.112}$$

The sum of the matrices $T^a T^a$ is proportional to the unit matrix, and is independent of the group, showing that the quantum potential is a Coulomb potential. This is confirmed by the fact that the quantum potential of the Abelian field is a Coulomb potential. The analytical confining potential derived above is a sum of a classical linear term and quantum Coulomb term, as expressed by Eq. (3.105) □

The above procedures also lead to the following theorem.

Theorem 3.28. *In pure Yang-Mills fields, the classical localized field as a vacuum produces the linear confinement potential between a charge and an anticharge, which are created by the pure Yang-Mills field. A non-zero binding energy between the paired charges generates a finite positive mass of the pure Yang-Mills field, namely, the mass gap, as shown in Eq. (3.85). Furthermore, a non-zero classical field gives rise to a local mass gap with non-zero positive value for the Yang-Mills field, as shown in Eq. (3.96). These masses give an answer to the mass gap problem of the Millennium problem.*

4 Conclusions

Despite usual mathematical papers contain no conclusion section, this section describes the conclusions. This paper presented a mathematical solution to the Yang-Mills existence and mass gap problem, which is one of the Millennium problems. The problem is widely known as a longstanding important problem. This paper has aimed at constructing solid mathematical foundations of field theory. The solution comprises two theorems derived via processes constructing associated sequential lemmas. For the first theorem, this paper formulated mathematically well-defined pure Yang-Mills fields, and two cutoffs were introduced. The first cutoff corresponds physically to the cutoff of the ultraviolet divergence, whose dominant contribution is caused by quadratic self-energies associated with Higgs fields in the relatively low-energy regime. With the aid of the following relativistic cutoff, ultraviolet divergences are avoided. In this theory, the four-dimensional space-time continuum is divided into arbitrarily shaped spacetime elements. The spacetime element includes space-like hypersurfaces, which is normal to the time axis. Fields are expanded in terms of scalar plane-waves, which are Poincar?/Lorentz covariant/invariant. The expansion coefficients have a symmetry represented by a compact and simple group. Below the first cutoff energy, which is lower than the second cutoff energy corresponding physically to the Planck energy, the momentum is continuous. Due to the local periodicity of space-time elements without long-range order, a new effective field with discrete momentums can be introduced. By interactions with the above effective field, quantum particles of gluons are only rarely excited beyond the first cutoff energy. The second cutoff at the high energy, which corresponds physically to the Planck energy, prevents unphysical losses of intermediate states into black holes. Quantities, which correspond physically to vacuum expectation values for operators, have been analyzed within the regime below the first cutoff energy. Corresponding mathematical formulations constructed from axioms have a basis in the traditional physical field theory except for high energies. The first theorem for this Poincar?/Lorentz covariant/invariant formalism without ultraviolet divergences presents an answer to the existence problem. Next, the pure Yang-Mills field considered is composed of classical stationary and quantum fluctuation fields. The classical field, which is a vacuum, has localized and unlocalized fields. From the Wilson loop for the classical localized field, there is a linear potential between a charge and an anticharge, and the binding energy yields a mass gap, by stabilizing the system compared to the Coulomb phase. Meanwhile, quantum fluctuations add a Coulomb potential to the linear potential. In the action, a local mass term appears due to the classical field. Due to these masses, the second theorem presents an answer to the mass gap problem. Consequently, the Yang-Mills existence and mass gap problem has been solved.

Acknowledgments

The author is grateful to Emeritus Prof. Hikaru Sato of Hyogo University of Education for variable discussions. The author also would like to thank a professional English editing/proofreading company for proofreading the manuscript by a native speaker.

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