BILATERAL LERCH THETA AND THETA STAR FUNCTION AND QUADRILATERAL LERCH ZETA AND ZETA STAR FUNCTIONS

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Abstract. In the present paper, we construct theta functions with two parameters $a, b \in \mathbb{R}$ which satisfy Jacobi's modular relation. Moreover, we give zeta functions with two parameters $a, b \in \mathbb{R}$ which satisfy Riemann's functional equation by the theta functions with two parameters.

1. Introduction

1.1. **Theta functions.** We review some of the standard facts on theta and related functions. Define the theta function by

$$\theta(v) := \sum_{n=-\infty}^{\infty} \exp\bigl(-\pi v n^2\bigr), \qquad v > 0$$

(e.g., [5, (2.4.9)]). It is widely-known that $\theta(v)$ satisfies Jacobi's modular relation

$$\theta(v) = v^{-1/2}\theta(v^{-1}) \tag{1.1}$$

(e.g., [5, (2.4.10)]). In [5, Problems in Chapter 2.4], the functions

$$\theta_1(a,v) := \sum_{n=-\infty}^{\infty} \exp\bigl(-\pi v (n+a)^2\bigr), \qquad \theta_2(a,v) := \sum_{n=-\infty}^{\infty} \exp\bigl(-\pi v n^2 + 2\pi i n a\bigr),$$

$$\theta_3(a,v) := \sum_{n=-\infty}^{\infty} (n+a)\exp\bigl(-\pi v (n+a)^2\bigr), \qquad \theta_4(a,v) := \sum_{n=-\infty}^{\infty} n\exp\bigl(-\pi v n^2 + 2\pi i n a\bigr),$$

are defined as generalizations of $\theta(v)$. These functions satisfy

$$\theta_1(v,a) = v^{-1/2}\theta_2(v^{-1},a), \qquad \theta_3(v,a) = v^{-3/2}\theta_4(v^{-1},a)$$

(see [5, Problems 2.4.3 and 2.4.5]). Furthermore, when $a, b, v > 0$, one has (see [1, (2.1)])

$$\sqrt{v}\sum_{n=-\infty}^{\infty} \exp\bigl(-\pi(n+a)^2 v + 2\pi i (n+a) b\bigr) = \sum_{n=-\infty}^{\infty} \exp\bigl(-v^{-1}\pi (n+b)^2 - 2\pi i n a\bigr). \tag{1.2}$$

It is well-known that, roughly speaking, the Riemann zeta function $\zeta(s)$ is the Mellin transform of the theta function $\theta(v)$. More precisely, we have

$$\pi^{-s/2}\Gamma\Bigl(\frac{s}{2}\Bigr)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \bigl(u^s + u^{1-s}\bigr)\bigl(\theta(u^2)-1\bigr)\frac{du}{2u} \tag{1.3}$$

(see [4, (1.3.5)] or [11, Chapter 2.6]).

2010 *Mathematics Subject Classification.* Primary 11M35, Secondary 11M26.
Key words and phrases. functional equation, modular relation, theta functions, zeta functions.

1.2. **Zeta functions.** In this subsection, we discuss zeta functions and their functional equations. As a generalization of $\zeta(s)$, define the Lerch zeta function $L(s,a,b)$ by

$$L(s,a,b) := \sum_{n=0}^{\infty} \frac{e^{2\pi inb}}{(n+a)^s}, \qquad \sigma > 1, \quad 0 < a, b \le 1.$$

The Hurwitz zeta function $\zeta(s,a)$ and the periodic zeta function $F(s,a)$ are defined as

$$\zeta(s,a) := L(s,a,1), \qquad F(s,a) := e^{-2\pi ia} L(s,1,a),$$

respectively. The Dirichlet series of $L(s,a,b)$ converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. Moreover, $L(s,a,1)$ has analytic continuation to $\mathbb{C}$ except $s = 1$, where there is a simple pole with residue 1 (e.g., [2, Chapter 12]). On the other hand, the Dirichlet series of the function $L(s,a,b)$ with $0 < b < 1$ converges uniformly in each compact subset of the half-plane $\sigma > 0$ (e.g., [6, p. 20]). Furthermore, the function $L(s,a,b)$ with $0 < b < 1$ is analytically continuable to the whole complex plane (e.g., [6, Chapter 2.2]). Note that one has $\zeta(s,1) = F(s,1) = \zeta(s)$.

For simplicity, we put

$$\Gamma_\pi(s) := \frac{\Gamma(s)}{(2\pi)^s}, \qquad \Gamma_{\cos}(s) := 2\Gamma_\pi(s) \cos\Bigl(\frac{\pi s}{2}\Bigr), \qquad \Gamma_{\sin}(s) := 2\Gamma_\pi(s) \sin\Bigl(\frac{\pi s}{2}\Bigr).$$

Then, the Riemann zeta function $\zeta(s)$ satisfies Riemann's functional equation

$$\zeta(1-s) = \Gamma_{\cos}(s)\zeta(s). \tag{1.4}$$

The functional equation for $\zeta(s,a)$ and $F(s,a)$ are expressed as

$$F(1-s,a) = \Gamma_\pi(s)\bigl(e^{\pi is/2}\zeta(s,a) + e^{-\pi is/2}\zeta(s,1-a)\bigr), \quad 0 < a < 1, \tag{1.5}$$

(e.g., [2, Exercises 12.2]). Moreover, the functional equation for $L(s,a,b)$ are given by

$$L(1-s,a,b) = \Gamma_\pi(s)\Bigl(e^{\pi is/2-2\pi iab}L(s,b,-a) + e^{-\pi is/2+2\pi ia(1-b)}L(s,1-b,a)\Bigr) \tag{1.6}$$

when $0 < b < 1$ (e.g., [6, Theorem 2.3.2]). It should be noted that the gamma factors of the functional equations in (1.4) and (1.5) do not depend on $0 < a < 1$ but the gamma factor of the functional equation (1.6) contains $e^{-2\pi iab}$ and $e^{2\pi ia(1-b)}$.

We can see that the functional equation (1.4) is much simpler than (1.5) and (1.6). In order to construct a zeta function satisfying Riemann's functional equation (1.4), for $0 < a \le 1/2$, we define the quadrilateral zeta function $Q(s,a)$ as

$$2Q(s,a) := \zeta(s,a) + \zeta(s,1-a) + F(s,a) + F(s,1-a). \tag{1.7}$$

Based on the facts mentioned above, the function $Q(s,a)$ can be continued analytically to the whole complex plane except $s = 1$. In [8, Theorem 1.1], it is shown that

$$Q(1-s,a) = \Gamma_{\cos}(s)Q(s,a), \qquad 0 < a \le 1/2. \tag{1.8}$$

It should be noted that (1.8) does not contradict to Hamburger's theorem [3, Staz 1] (see also [8, Section 1.3]). Moreover, this function has the following properties (see [8, Theorem 1.2] and [9, Theorem 1.1]).

- For any $0 < a \le 1/2$, there exist positive constants $A(a)$ and $T_0(a)$ such that the number of zeros of $Q(s,a)$ on the line segment from $1/2$ to $1/2 + iT$ is greater than $A(a)T$ whenever $T \ge T_0(a)$.
- There exists $a_0 = 0.1183751396...$ such that
  (1) $Q(\sigma,a_0)$ has a unique double real zero at $\sigma = 1/2$ when $\sigma \in (0,1)$,
  (2) for any $a \in (a_0, 1/2]$, the function $Q(\sigma,a)$ has no real zero in $\sigma \in (0,1)$,
  (3) for any $a \in (0,a_0)$, $Q(\sigma,a)$ has at least two real zeros in $\sigma \in (0,1)$.

2. Main results

This paper has the following two aims.

- We construct theta functions with two parameters $a, b \in \mathbb{R}$ which satisfy Jacobi's modular relation (1.1) in Theorem 2.3.
- We construct zeta functions with two parameters $a, b \in \mathbb{R}$ which satisfy Riemann's functional equation (1.4) in Theorem 2.4.

Moreover, we discuss quasi-commutativity of these parametrise $a, b \in \mathbb{R}$, Fourier expansions and relations between these theta and zeta function via integral representations. Note that theta functions with one parameter $0 < a < 1/2$ satisfying Jacobi's modular relation (1.1) and zeta functions with one parameter $0 < a < 1/2$ satisfying Riemann's functional equation (1.1) have been already given in [8, (2.1)] and [8, (1.2)], respectively.

The contents of the paper are as follows. In Section 2.1, we recall the modular relation and give new results of the theta function $G(u, a)$ and the zeta function $Q(s, a)$ introduced in [8, Sections 2.1 and 1.1]. In Section 2.2, we give theta functions $G_Q(u, a, b)$ which satisfy Jacobi's modular relation (1.1) and show that $G_Q(u, a, b)$ have periodicities, quasi-periodicities, symmetry or skew-symmetry and so on (see Theorem 2.3). Furthermore, we construct zeta functions $Q(s, a, b)$ which satisfy Riemann's functional equation (1.4) and other properties mentioned above by using $G_Q(u, a, b)$ (see Theorem 2.4). Moreover, we prove that functions $G_X(u, a, b)$ and $X(s, a, b)$ defined in Section 2.2 have similar properties. In Section 3, we prove Proposition 2.1 and Theorem 2.3. Section 4 is devoted to the proof of Proposition 2.2. In Section 5, we show Theorem 2.4.

2.1. Theta and zeta functions with one parameter.

We first recall the modular relation and give new results on the theta function $G(u, a)$. For $u > 0$ and $a \in \mathbb{R}$, define the functions

$$G_Q(u, a) := G_Z(u, a) + G_P(u, a), \qquad G_X(u, a) := G_Y(u, a) + G_O(u, a),$$

where $G_Z(u, a)$ $G_P(u, a)$, $G_Y(u, a)$ and $G_O(u, a)$ are given as

$$G_Z(u, a) := \sum_{n \in \mathbb{Z}} \exp\bigl(-\pi u^2 (n+a)^2\bigr), \qquad G_P(u, a) := \sum_{n \in \mathbb{Z}} \exp\bigl(-\pi u^2 n^2 - 2\pi i n a\bigr).$$

$$G_Y(u, a) := \sum_{n \in \mathbb{Z}} (n+a) \exp\bigl(-\pi u^2 (n+a)^2\bigr), \qquad G_O(u, a) := i \sum_{n \in \mathbb{Z}} n \exp\bigl(-\pi u^2 n^2 - 2\pi i n a\bigr),$$

respectively. Note that the first equality in (2.4) has already shown in [8, (2.1)] when $0 < a < 1/2$.

Proposition 2.1. *We have the five statements below;*

(**1**) Special cases. *When $a \in \mathbb{Z}$.*

$$G_Q(u, a) = 2\theta(u^2), \qquad G_X(u, a) = 0. \tag{2.1}$$

(**2**) Periodicity. *For $a \in \mathbb{R}$,*

$$G_Q(u, a) = G_Q(u, a+1), \qquad G_X(u, a) = G_X(u, a+1). \tag{2.2}$$

(**3**) Symmetry or skew-symmetry.

$$G_Q(u, a) = G_Q(u, -a), \quad G_X(u, a) = -G_X(u, -a). \tag{2.3}$$

(**4**) Modular relations.

$$G_Q(u, a) = u^{-1} G_Q(u^{-1}, a), \qquad G_X(u, a) = u^{-3} G_X(u^{-1}, a). \tag{2.4}$$

(**5**) Fourier expansions. *When* $a \in \mathbb{R} \setminus \mathbb{Z}$*,*

$$
\begin{aligned}
G_Q(u,a) &= 1 + \frac{1}{u} + \frac{2}{u} \sum_{n=1}^{\infty} \Bigl(u \exp\bigl(-\pi u^2 n^2\bigr) + \exp\bigl(-\pi u^{-2} n^2\bigr) \Bigr) \cos(2\pi n a), \\
G_X(s,a) &= 1 + \frac{1}{u^3} + \frac{2}{u^3} \sum_{n=1}^{\infty} n \Bigl(u^3 \exp\bigl(-\pi u^2 n^2\bigr) + \exp\bigl(-\pi u^{-2} n^2\bigr) \Bigr) \sin(2\pi n a).
\end{aligned}
\tag{2.5}
$$

We next recall the functional equation and show some new results on the zeta functions $Q(s,a)$ and $X(s,a)$. For $a \in \mathbb{R}$ and $\Re(s) > 1$, put

$$
2Q(s,a) := Z(s,a) + P(s,a), \qquad 2X(s,a) = Y(s,a) + O(s,a)
$$

where $Z(s,a)$, $P(s,a)$, $Y(s,a)$ and $O(s,a)$ are defined as

$$
\begin{aligned}
Z(s,a) &:= \sum_{n+a \ne 0} \frac{1}{|n+a|^s}, \qquad & P(s,a) &:= \sum_{0 \ne n \in \mathbb{Z}} \frac{e^{-2\pi i n a}}{|n|^s}, \\
Y(s,a) &:= \sum_{n+a \ne 0} \frac{\operatorname{sgn}(n+a)}{|n+a|^s}, \qquad & O(s,a) &:= i \sum_{0 \ne n \in \mathbb{Z}} \frac{\operatorname{sgn}(n) e^{-2\pi i n a}}{|n|^s},
\end{aligned}
$$

respectively. Note that $Z(s,a)$, $P(s,a)$, $Y(s,a)$, $O(s,a)$ $Q(s,a)$ and $X(s,a)$ with $0 < a < 1/2$ have already given in ([8, Section 1.1] and [10, Section 1.2]). Moreover, both functional equations in (2.10) have already given in ([8, (1.2)] and [10, (3.15)]) when $0 < a < 1/2$ (see Section 1.2). Thus, in this paper, we show the functional equations and other properties of $Q(s,a)$ and $X(s,a)$ for not only $0 < a < 1/2$ but also $a \in \mathbb{R}$.

Proposition 2.2. *We have the six statements below;*

(**0**) Integral representations. *For* $s \in \mathbb{C}$ *and* $a \in \mathbb{R} \setminus \mathbb{Z}$*,*

$$
\begin{aligned}
\pi^{-s/2} \Gamma\Bigl(\frac{s}{2}\Bigr) Q(s,a) &= \frac{1}{s(s-1)} + \int_1^{\infty} \bigl(u^s + u^{1-s}\bigr) \bigl(G_Q(u,a) - 1\bigr) \frac{du}{u}, \\
\pi^{-(s+1)/2} \Gamma\Bigl(\frac{s+1}{2}\Bigr) X(s,a) &= \int_1^{\infty} \bigl(u^s + u^{1-s}\bigr) G_X(u,a) du.
\end{aligned}
\tag{2.6}
$$

Note that the case $a \in \mathbb{Z}$ *is given in (1.3).*

(**1**) Special cases. *For* $a \in \mathbb{Z}$*,*

$$
Q(s,a) = 2\zeta(s), \quad X(s,a) = 0. \tag{2.7}
$$

(**2**) Periodicity. *For* $a \in \mathbb{R} \setminus \mathbb{Z}$*,*

$$
Q(s,a) = Q(s,a+1), \qquad X(s,a) = X(s,a+1). \tag{2.8}
$$

(**3**) Symmetry or skew-symmetry.

$$
Q(s,a) = Q(s,-a), \qquad X(s,a) = -X(s,-a). \tag{2.9}
$$

(**4**) Functional equations. *For* $s \in \mathbb{C}$*,*

$$
Q(1-s,a) = \Gamma_{\cos}(s) Q(s,a), \qquad X(1-s,a) = \Gamma_{\sin}(s) X(s,a). \tag{2.10}
$$

(**5**) Fourier expansions. *When* $a \in \mathbb{R} \setminus \mathbb{Z}$ *and* $0 < \Re(s) < 1$*,*

$$
\begin{aligned}
Q(s,a) &= \sum_{n=1}^{\infty} \Bigl(\frac{1}{n^s} + \frac{\Gamma_{\cos}(1-s)}{n^{1-s}} \Bigr) \cos(2\pi n a), \\
X(s,a) &= \sum_{n=1}^{\infty} \Bigl(\frac{1}{n^s} + \frac{\Gamma_{\sin}(1-s)}{n^{1-s}} \Bigr) \sin(2\pi n a).
\end{aligned}
\tag{2.11}
$$

Remark. When $\Re(s)>1$ is fixed, one has $\int_0^1 a^{-s}da$, $\int_0^1(1-a)^{-s} \notin L^1[0,1]$ and

$$\int_0^1 \bigl(2Q(s,a)-a^{-s}-(1-a)^{-s}\bigr)da \in L^1[0,1].$$

Hence, the Fourier coefficient

$$\int_0^1 Q(s,a)e^{-2\pi ina}da, \qquad n\in\mathbb{Z}$$

does not converge for $\Re(s)>1$. By the functional equation (2.10), the Fourier coefficient above does not converge for $\Re(s)<0$. Similarly, the Fourier coefficient

$$\int_0^1 X(s,a)e^{-2\pi ina}da, \qquad n\in\mathbb{Z}$$

does not converge for $\Re(s)>1$ or $\Re(s)<0$. Thus, we have Fourier expansions of $Q(s,a)$ and $X(s,a)$ for only $0<\Re(s)<1$.

2.2. **Theta and zeta functions with two parameters.** In this subsection, we state the two main results in the present paper. First, we give theta functions with two parameters $a,b\in\mathbb{R}$ which satisfy Jacobi's modular relation (1.1). For $a,b\in\mathbb{R}$, put

$$G_Q(u,a,b) := G_Z(u,a,b)+G_P(u,a,b), \qquad G_X(u,a,b) := G_Y(u,a,b)+G_O(u,a,b),$$

where $G_Z(u,a,b)$ $G_P(u,a,b)$, $G_Y(u,a,b)$ and $G_O(u,a,b)$ are defined as

$$\begin{aligned} G_Z(u,a,b) &:= \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^2(n+a)^2+2\pi i(n+a)b\bigr),\\ G_P(u,a,b) &:= \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^2(n+b)^2-2\pi ina\bigr),\end{aligned}$$

$$\begin{aligned} G_Y(u,a,b) &:= \sum_{n\in\mathbb{Z}}(n+a) \exp\bigl(-\pi u^2(n+a)^2+2\pi i(n+a)b\bigr),\\ G_O(u,a,b) &:= i\sum_{n\in\mathbb{Z}}(n+b) \exp\bigl(-\pi u^2(n+b)^2-2\pi ina\bigr).\end{aligned}$$

We name $G_Q(u,a,b)$ and $G_X(u,a,b)$ bilateral Lerch theta function and bilateral Lerch theta star function, respectively. As a generalization of Proposition 2.1, we have the following.

Theorem 2.3. *We have the six statements below;*

(**1**) Special cases.

$$\begin{aligned} G_Q(u,a,0) = G_Q(u,a), &\qquad G_X(u,a,0) = G_X(u,a),\\ G_Q(u,0,b) = G_Q(u,b), &\qquad G_X(u,0,b) = iG_X(u,b).\end{aligned} \tag{2.12}$$

(**2**) Periodicity and quasi-periodicity.

$$\begin{aligned} G_Q(u,a,b) = G_Q(u,a+1,b), &\qquad G_X(u,a,b) = G_X(u,a+1,b),\\ G_Q(u,a,b+1) = e^{2\pi ia}G_Q(u,a,b), &\qquad G_X(u,a,b+1) = e^{2\pi ia}G_X(u,a,b).\end{aligned} \tag{2.13}$$

(**3**) Symmetry or skew-symmetry.

$$G_Q(u,a,-b) = G_Q(u,-a,b), \qquad G_X(s,a,-b) = -G_X(s,-a,b). \tag{2.14}$$

(**4**) Modular relations.

$$G_Q(u,a,b) = u^{-1}G_Q(u^{-1},a,b), \qquad G_X(u,a,b) = u^{-3}G_X(u^{-1},a,b). \tag{2.15}$$

(**5**) Fourier expansions. *When* $a \in \mathbb{R} \setminus \mathbb{Z}$,

$$\begin{aligned} G_Q(u,a,b) &= \frac{1}{u} \sum_{n \in \mathbb{Z}} \Big(u \exp\big(-\pi u^2 (n-b)^2\big) + \exp\big(-\pi u^{-2} (n-b)^2\big) \Big) e^{2\pi i n a}, \\ G_Y(u,a,b) &= \frac{-i}{u^3} \sum_{n \in \mathbb{Z}} (n-b) \Big(u^3 \exp\big(-\pi u^2 (n-b)^2\big) + \exp\big(-\pi u^{-2} (n-b)^2\big) \Big) e^{2\pi i n a}. \end{aligned} \tag{2.16}$$

(**6**) Quasi-commutativity of the second and third variables.

$$G_Q(u,-b,a) = e^{-2\pi i a b} G_Q(u,a,b), \qquad G_X(u,-b,a) = i e^{-2\pi i a b} G_X(u,a,b). \tag{2.17}$$

Our next goal is to construct zeta functions with two parameters $a, b \in \mathbb{R}$ which satisfy Riemann's functional equation (1.4). For $a, b \in \mathbb{R}$ and $\Re(s) > 1$, put

$$Q(s,a,b) = Z(s,a,b) + P(s,a,b), \qquad X(s,a,b) = Y(s,a,b) + O(s,a,b),$$

where $Z(s,a,b)$, $P(s,a,b)$, $Y(s,a,b)$ and $O(s,a,b)$ are defined as

$$\begin{aligned} Z(s,a,b) &:= \sum_{n+a \neq 0} \frac{e^{2\pi i (n+a) b}}{|n+a|^s}, \qquad & P(s,a,b) &:= \sum_{n+b \neq 0} \frac{e^{-2\pi i n a}}{|n+b|^s}, \\ Y(s,a,b) &:= \sum_{n+a \neq 0} \frac{\operatorname{sgn}(n+a) e^{2\pi i (n+a) b}}{|n+a|^s}, \qquad & O(s,a,b) &:= i \sum_{n+b \neq 0} \frac{\operatorname{sgn}(n+b) e^{-2\pi i n a}}{|n+b|^s}, \end{aligned}$$

respectively. We call $Q(u,a,b)$ and $X(u,a,b)$ quadrilateral Lerch zeta function and quadrilateral Lerch zeta star function, respectively (see (5.3), (5.4), (5.7) and (5.8)). Note that some functions related to $L(s,a,b)$ are define in [7, (2.2) and (2.3)] and their functional equations, whose Gamma factors depend on the parameters $a, b \in (0,1)$, are proved in [7, Theorem 2.1]. The next theorem is a generalization of Proposition 2.2.

Theorem 2.4. *We have the following seven statements;*

(**0**) Integral representations. *For* $s \in \mathbb{C}$, $a, b \in \mathbb{R} \setminus \mathbb{Z}$,

$$\begin{aligned} \pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big) Q(s,a,b) &= \int_1^\infty (u^s + u^{1-s}) G_Q(u,a,b) \frac{du}{u}, \\ \pi^{-(s+1)/2} \Gamma\Big(\frac{s+1}{2}\Big) X(s,a,b) &= \int_1^\infty (u^s + u^{1-s}) G_X(u,a,b) du. \end{aligned} \tag{2.18}$$

Note that the case $a \in \mathbb{Z}$ *or* $b \in \mathbb{Z}$ *is given in (2.6).*

(**1**) Special cases.

$$\begin{aligned} Q(s,a,0) &= Q(s,a), \qquad & X(s,a,0) &= X(s,a), \\ Q(s,0,b) &= Q(s,b), \qquad & X(s,0,b) &= iX(s,b). \end{aligned} \tag{2.19}$$

(**2**) Periodicity and quasi-periodicity.

$$\begin{aligned} Q(s,a,b) &= Q(s,a+1,b), \qquad & X(s,a,b) &= X(s,a+1,b), \\ Q(s,a,b+1) &= e^{2\pi i a} Q(s,a,b), \qquad & X(s,a,b+1) &= e^{2\pi i a} X(s,a,b). \end{aligned} \tag{2.20}$$

(**3**) Symmetry or skew-symmetry.

$$Q(s,a,-b) = Q(s,-a,b), \qquad X(s,a,-b) = -X(s,-a,b). \tag{2.21}$$

(**4**) Functional equations.

$$Q(1-s,a,b) = \Gamma_{\cos}(s) Q(s,a,b), \qquad X(1-s,a,b) = \Gamma_{\sin}(s) X(s,a,b). \tag{2.22}$$

(**5**) Fourier expansions. *When* $a, b \in \mathbb{R} \setminus \mathbb{Z}$ *and* $0 < \Re(s) < 1$,

$$\begin{aligned} Q(s,a,b) &= \frac{1}{2} \sum_{n-b\neq 0} \left(\frac{1}{|n-b|^s} + \frac{\Gamma_{\cos}(1-s)}{|n-b|^{1-s}} \right) e^{2\pi i n a}, \\ X(s,a,b) &= \frac{1}{2i} \sum_{n-b\neq 0} \left(\frac{\operatorname{sgn}(n-b)}{|n-b|^s} + \frac{\Gamma_{\sin}(1-s)\operatorname{sgn}(n-b)}{|n-b|^{1-s}} \right) e^{2\pi i n a}. \end{aligned} \tag{2.23}$$

(**6**) Quasi-commutativity of the second and third variables.

$$Q(s,-b,a) = e^{-2\pi i a b} Q(s,a,b), \qquad X(s,b,-a) = i e^{-2\pi i a b} X(s,a,b). \tag{2.24}$$

3. Proofs of Proposition 2.1 and Theorem 2.3

Proof of Proposition 2.1. We have the first equation in (2.1) by $G_Z(u,a) = G_P(u,a) = \theta(u^2)$ when $a \in \mathbb{Z}$. The second equation in (2.1) is shown by $G_Y(u,a) = G_O(u,a) = 0$ if $a \in \mathbb{Z}$. The definitions of $G_Q(u,a)$ and $G_X(u,a)$ imply the second statement of Proposition 2.1. We can easily show the third statement from $G_Z(u,a) = G_Z(u,-a)$, $G_P(u,a) = G_P(u,-a)$, $G_Y(u,a) = -G_Y(u,-a)$ and $G_O(u,a) = -G_O(u,-a)$.

For $a, u > 0$, it is widely known that (see [4, p. 13, (6)])

$$G_Z(u,a) = u^{-1} G_P(u^{-1},a), \qquad G_P(u,a) = u^{-1} G_Z(u^{-1},a). \tag{3.1}$$

Hence, we have the first equation in (2.4). From the definitions, one has

$$\frac{\partial}{\partial a} G_Z(u,a) = -2\pi u^2 G_Y(u,a), \qquad \frac{\partial}{\partial a} G_P(u,a) = -2\pi G_O(u,a). \tag{3.2}$$

Thus, by using (3.1), we have

$$-2\pi u^2 G_Y(u,a) = \frac{\partial}{\partial a} G_Z(u,a) = \frac{\partial}{\partial a} u^{-1} G_P(u^{-1},a) = -2\pi u^{-1} G_O(u^{-1},a),$$

$$-2\pi G_O(u,a) = \frac{\partial}{\partial a} G_P(u,a) = \frac{\partial}{\partial a} u^{-1} G_Z(u^{-1},a) = -2\pi u^{-3} G_Y(u^{-1},a).$$

Therefore, we obtain

$$G_Y(u,a) = u^{-3} G_O(u^{-1},a), \qquad G_O(u,a) = u^{-3} G_Y(u^{-1},a). \tag{3.3}$$

The equations above imply the second equation in (2.4).

From the definition of $G_Z(u,a)$, we can easily see that

$$G_P(u,a) = \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^2 n^2 - i2\pi n a\bigr) = 1 + 2\sum_{n=1}^{\infty} \exp\bigl(-\pi u^2 n^2\bigr) \cos(2\pi n a).$$

Moreover, by the first equation of (3.1), we have

$$G_Z(u,a) = u^{-1} G_P(u^{-1},a) = \frac{1}{u} + \frac{2}{u} \sum_{n=1}^{\infty} \exp\bigl(-\pi u^{-2} n^2\bigr) \cos(2\pi n a).$$

Hence we obtain the first equation of (2.5). Similarly, we have

$$G_O(u,a) = i\sum_{n\in\mathbb{Z}} n \exp\bigl(-\pi u^2 n^2 - i2\pi n a\bigr) = 1 + 2\sum_{n=1}^{\infty} n\exp\bigl(-\pi u^2 n^2\bigr) \sin(2\pi n a),$$

$$G_Y(u,a) = u^{-3} G_O(u^{-1},a) = \frac{1}{u^3} + \frac{2}{u^3} \sum_{n=1}^{\infty} n\exp\bigl(-\pi u^{-2} n^2\bigr) \sin(2\pi n a)$$

which implies the second equation of (2.5). □

Proof of Theorem 2.3. We can easily show the first, second and third equations in (2.12). The fourth equation is proved by

$$G_Y(u,0,b) = -iG_O(u,-b) = iG_O(u,b) \quad \text{and} \quad G_O(u,0,b) = iG_Y(u,b).$$

The first and second equations in (2.13) are trivial. The third formula is shown by

$$G_Z(u,a,b+1) = \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^2(n+a)^2 + 2\pi i(n+a)(b+1)\bigr) = e^{2\pi ia} G_Z(u,a,b),$$

$$\begin{aligned} G_P(u,a,b+1) &= \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^2(n+b+1)^2 - 2\pi ina\bigr) \\ &= \sum_{m\in\mathbb{Z}} \exp\bigl(-\pi u^2(m+b)^2 - 2\pi i(m-1)a\bigr) = e^{2\pi ia} G_P(u,a,b). \end{aligned}$$

Similarly, we can prove formulas $G_Y(u,a,b+1) = e^{2\pi ia}G_Y(u,a,b)$ and $G_O(u,a,b+1) = e^{2\pi ia}G_O(u,a,b)$ which imply the fourth equation in (2.13).

The first formula in (2.14) is shown by

$$\begin{aligned} G_Z(u,a,-b) &= \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^2(n+a)^2 - 2\pi i(n+a)b\bigr) \\ &= \sum_{m\in\mathbb{Z}} \exp\bigl(-\pi u^2(m-a)^2 + 2\pi i(m-a)b\bigr) = G_Z(u,-a,b) \end{aligned}$$

and $G_P(u,a,-b) = G_P(u,-a,b)$, which is proved similarly. Moreover, we have

$$\begin{aligned} G_Y(u,a,-b) &= \sum_{n\in\mathbb{Z}} (n+a) \exp\bigl(-\pi u^2(n+a)^2 - 2\pi i(n+a)b\bigr) \\ &= -\sum_{m\in\mathbb{Z}} (m-a) \exp\bigl(-\pi u^2(m-a)^2 + 2\pi i(m-a)b\bigr) = -G_Y(u,-a,b) \end{aligned}$$

and $G_O(u,a,-b) = -G_O(u,-a,b)$ which imply the second equation in (2.14).

The equality (1.2) implies

$$G_Z(u,a,b) = u^{-1}G_P(u^{-1},a,b), \qquad G_P(u,a,b) = u^{-1}G_Z(u^{-1},a,b). \tag{3.4}$$

Thus, we immediately obtain the first equation in (2.15). Furthermore, one has

$$\frac{\partial}{\partial b} G_Z(u,a,b) = 2\pi i G_Y(u,a,b), \qquad \frac{\partial}{\partial b} G_P(u,a,b) = 2\pi i u^2 G_O(u,a,b). \tag{3.5}$$

By (3.4) and (3.5), we have

$$2\pi i G_Y(u,a,b) = \frac{\partial}{\partial b} G_Z(u,a,b) = \frac{\partial}{\partial b} u^{-1} G_P(u^{-1},a,b) = 2\pi i u^{-3} G_O(u^{-1},a,b),$$

$$2\pi i u^2 G_O(u,a,b) = \frac{\partial}{\partial a} G_P(u,a,b) = \frac{\partial}{\partial a} u^{-1} G_Z(u^{-1},a,b) = 2\pi i u^{-1} G_Y(u^{-1},a,b),$$

which imply

$$G_Y(u,a,b) = u^{-3}G_O(u^{-3},a,b), \qquad G_O(u,a,b) = u^{-3}G_Y(u^{-1},a,b). \tag{3.6}$$

Therefore, we have the second equation in (2.15).

From (3.4), it holds that

$$G_Z(u,a,b) = u^{-1} \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^{-2}(n+b)^2 - 2\pi ina\bigr) = u^{-1} \sum_{m\in\mathbb{Z}} \exp\bigl(-\pi u^{-2}(m-b)^2\bigr) e^{2\pi ima}.$$

Hence we have the first Fourier expansion of (2.16). By (3.6), we have

$$G_Y(u,a,b) = -iu^{-3} \sum_{n\in\mathbb{Z}} (n-b) \exp\bigl(-\pi u^{-2}(n-b)^2 + 2\pi ina\bigr).$$

Therefore, we obtain the second equation in (2.16).

By the definitions of $G_Z(u,a,b)$ and $G_P(u,a,b)$, it holds that

$$G_Z(u,-b,a) = e^{-2\pi iab} G_P(u,a,b) \tag{3.7}$$

since one has

$$\begin{aligned} G_Z(u,-b,a) &= \sum_{n\in\mathbb{Z}} \exp\bigl(-\pi u^2(n-b)^2 + 2\pi i(n-b)a\bigr) \\ &= \sum_{m\in\mathbb{Z}} \exp\bigl(-\pi u^2(m+b)^2 - 2\pi i(m+b)a\bigr) = e^{-2\pi iab} G_P(u,a,b). \end{aligned}$$

Changing variables $-b \to a$ and $a \to b$ in (3.7), we have $G_P(u,b,-a) = e^{-2\pi iab} G_Z(u,a,b)$. Applying the first equation of (2.14) to this formula, we obtain

$$G_P(u,-b,a) = e^{-2\pi iab} G_Z(u,a,b). \tag{3.8}$$

The relations (3.7) and (3.8) imply the first formula in (2.17). Moreover, one has

$$G_Y(u,-b,a) = ie^{-2\pi iab} G_O(u,a,b) \tag{3.9}$$

because we have

$$\begin{aligned} G_Y(u,-b,a) &= \sum_{n\in\mathbb{Z}} (n-b) \exp\bigl(-\pi u^2(n-b)^2 + 2\pi i(n-b)a\bigr) \\ &= -\sum_{m\in\mathbb{Z}} (m+b) \exp\bigl(-\pi u^2(m+b)^2 - 2\pi i(m+b)a\bigr) = ie^{-2\pi iab} G_O(u,a,b). \end{aligned}$$

Replacing variables $-b \to a$ and $a \to b$ in the equation (3.9), we obtain $G_O(u,b,-a) = -ie^{-2\pi iab} G_Y(u,a,b)$. Hence we have

$$G_O(u,-b,a) = ie^{-2\pi iab} G_Y(u,a,b) \tag{3.10}$$

from the relation $G_O(u,-b,a) = -G_O(u,b,-a)$. Clearly, the equations (3.9) and (3.10) imply the second formula in (2.17). □

4. Proof of Proposition 2.2

We can easily show (1), (2) and (3) of Proposition 2.2 by the definitions of $Q(s,a)$ and $X(s,a)$. Moreover, we have

$$P(s,a) = 2\sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s}, \qquad O(s,a) = 2\sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s}$$

when $a \in \mathbb{R} \setminus \mathbb{Z}$ and $0 < \Re(s) < 1$. The functional equation (1.5) implies

$$Z(1-s) = \Gamma_{\cos}(s) P(s,a), \qquad Y(1-s) = \Gamma_{\sin}(s) O(s,a)$$

(see also [10, (4.9) and (3.9)]). Hence, we have the Fourier expansions in (2.11) by the functional equations above, namely,

$$Z(s) = \Gamma_{\cos}(1-s) P(1-s,a) \quad \text{and} \quad Y(s) = \Gamma_{\sin}(1-s) O(1-s,a).$$

The functional equations in Proposition 2.2 are easily proved by the integral representations in (2.6). Hence, we show Proposition 2.2 (0).

Proof of (2.6) for $Q(s,a)$. Let $0 < a < 1$ and $\Re(s) > 1$. Then we have

$$2\int_0^\infty u^{s-1} G_Z(u,a) du = 2\sum_{n=0}^{\infty} \int_0^\infty u^{s-1} e^{-\pi u^2(n+a)^2} du + 2\sum_{n=0}^{\infty} \int_0^\infty u^{s-1} e^{-\pi u^2(n+1-a)^2} du.$$

The first infinite series can be rewritten as

$$\sum_{n=0}^{\infty} \int_0^\infty e^{-v} \left(\frac{v/\pi}{(n+a)^2}\right)^{s/2-1} \frac{dv/\pi}{(n+a)^2} = \pi^{-s/2}\Gamma\Bigl(\frac{s}{2}\Bigr) \sum_{n=0}^{\infty} (n+a)^{-s}.$$

Hence, we obtain

$$2\int_0^\infty u^{s-1} G_Z(u,a)du = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Z(s,a). \tag{4.1}$$

Similarly, when $\Re(s) > 1$, one has

$$2\int_0^\infty u^{s-1} \sum_{0\neq n\in\mathbb{Z}} \exp\big(-\pi u^2 n^2 - i2\pi na\big)du$$
$$= 2\sum_{n=1}^\infty \int_0^\infty u^{s-1}e^{-2\pi ina-\pi u^2n^2}du + 2\sum_{n=1}^\infty \int_0^\infty u^{s-1}e^{2\pi ina-\pi u^2n^2}.$$

Note that the second infinite series can be expressed as

$$\sum_{n=1}^\infty e^{2\pi ina}\int_0^\infty e^{-v}\left(\frac{v/\pi}{n^2}\right)^{s/2-1}\frac{dv/\pi}{n^2} = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\sum_{n=1}^\infty \frac{e^{2\pi ina}}{n^s}.$$

Thus, it holds that

$$2\int_0^\infty u^{s-1}\big(G_P(u,a)-1\big)du = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)P(s,a). \tag{4.2}$$

Therefore, when $\Re(s) > 1$, one has

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Q(s,a) = \int_0^\infty u^{s-1}\big(G_Q(u,a)-1\big)du. \tag{4.3}$$

Note that (4.3) with $0 < a < 1/2$ has already given in the proof of [8, Proposition 2.1]. By using the first equation in (2.4) and changing the variable $u \to v^{-1}$, we obtain

$$\int_0^1 u^{s-1}\big(G_Q(u,a)-1\big)du = \int_1^\infty v^{1-s}\Big(G_Q(v^{-1},a)-1\Big)\frac{dv}{v^2} = \int_1^\infty v^{1-s}\Big(G_Q(v,a)-v^{-1}\Big)\frac{dv}{v}$$

when $\Re(s) > 1$. Moreover, we have

$$\int_1^\infty v^{1-s}\frac{dv}{v} = \frac{1}{s-1}, \qquad \int_1^\infty v^{1-s}v^{-1}\frac{dv}{v} = \frac{1}{s}$$

if $\Re(s) > 1$. Thus we can easily see that

$$\int_0^1 u^{s-1}\big(G_Q(u,a)-1\big)du = \int_1^\infty v^{1-s}\big(G_Q(v,a)-1\big)\frac{dv}{v} + \frac{1}{s-1} - \frac{1}{s}.$$

Therefore, from (4.3) and the equation above, we have

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Q(s,a) = \int_0^1 u^{s-1}\big(G_Q(u,a)-1\big)du + \int_1^\infty u^{s-1}\big(G_Q(u,a)-1\big)du$$
$$= \frac{1}{s(s-1)} + \int_1^\infty u^{s-1}\big(G_Q(u,a)-1\big)du + \int_1^\infty u^{-s}\big(G_Q(u,a)-1\big)du$$

for $\Re(s) > 1$. The integrals above converge absolutely for all $s \in \mathbb{C}$, and so the formula holds, by analytic continuation, for all $s \in \mathbb{C}$. Hence we obtain the first equality in (2.6) for all $0 < a < 1$ and $s \in \mathbb{C}$. By the periodicities $G_Q(u,a)$ and $Q(s,a)$, the first equation in (2.6) holds for all $a \in \mathbb{R}\setminus\mathbb{Z}$. □

Proof of (2.6) for $X(s,a)$. Suppose $a \in \mathbb{R}\setminus\mathbb{Z}$. By (4.1) and (4.2), we have

$$\frac{\partial}{\partial a}\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Z(s,a) = \frac{\partial}{\partial a}\int_0^\infty u^{s-1}G_Z(u,a)du,$$
$$\frac{\partial}{\partial a}\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)P(s,a) = \frac{\partial}{\partial a}\int_0^\infty u^{s-1}\big(G_P(u,a)-1\big)du$$

when $\Re(s) > 1$. First, we consider the right-hands sides of the equalities above. From (3.2), we obtain

$$\frac{\partial}{\partial a}\int_0^\infty u^{s-1}G_Z(u,a)du = -2\pi\int_0^\infty u^{s+1}G_Y(u,a)du,$$
$$\frac{\partial}{\partial a}\int_0^\infty u^{s-1}\big(G_P(u,a)-1\big)du = -2\pi\int_0^\infty u^{s-1}G_O(u,a)du.$$

On the other hand, we can easily see that

$$\frac{\partial}{\partial a}Z(s,a) = \sum_{n+a\ne 0}\frac{\partial}{\partial a}\frac{1}{|n+a|^s} = -s\sum_{n+a\ne 0}\frac{\operatorname{sgn}(n+a)}{|n+a|^{s+1}} = -sY(s+1,a),$$

$$\frac{\partial}{\partial a}P(s,a) = \sum_{0\ne n\in\mathbb{Z}}\frac{\partial}{\partial a}\frac{e^{-2\pi ina}}{|n|^s} = -2\pi i\sum_{0\ne n\in\mathbb{Z}}\frac{\operatorname{sgn}(n)e^{-2\pi ina}}{|n|^{s-1}} = -2\pi O(s-1,a)$$

if $\Re(s) > 1$ is sufficiently large. By these formulas, we have

$$\frac{\partial}{\partial a}\Gamma\Big(\frac{s}{2}\Big)Z(s,a) = \Gamma\Big(\frac{s}{2}\Big)(-s)Y(s+1,a) = -2\Gamma\Big(\frac{s+2}{2}\Big)Y(s+1,a),$$
$$\frac{\partial}{\partial a}\Gamma\Big(\frac{s}{2}\Big)P(s,a) = -2\pi\Gamma\Big(\frac{s}{2}\Big)O(s-1,a)$$

for all $s\in\mathbb{C}$. Therefore, when $\Re(s) > 1$, we obtain

$$\int_0^\infty u^sG_Y(u,a)du = \pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)Y(s,a),$$
$$\int_0^\infty u^sG_O(u,a)du = \pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)O(s,a).$$

The equations above imply

$$\pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)X(s,a) = \int_0^\infty u^sG_X(u,a)du.$$

Applying the second modular relation in (2.4), we obtain

$$\pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)X(s,a) = \int_0^1 u^sG_X(u,a)du + \int_1^\infty u^sG_X(u,a)du$$
$$= \int_1^\infty u^sG_X(u,a)du + \int_1^\infty v^{-s}G_X(v^{-1},a)\frac{dv}{v^2} = \int_1^\infty \big(u^s+u^{1-s}\big)G_X(u,a)du.$$

Obviously the last integral converges absolutely for all $s\in\mathbb{C}$. Thus formula above holds for all $s\in\mathbb{C}$ by analytic continuation. Hence we obtain the second equality in (2.6) for all $a\in\mathbb{R}\setminus\mathbb{Z}$ and $s\in\mathbb{C}$. □

5. Proof of Theorem 2.4

The functional equations in (2.22) are proved by the integral representations (2.18) since the right-hand side is unchanged if s is replaced by $1-s$. However, we give proofs of equations in (2.22) by using the functional equation (1.6) of the Lerch zeta function.

Proof of (2.22). Let us suppose that $0 < a, b < 1$. Clearly, the functional equation (1.6) can be rewritten as

$$e^{2\pi iab}L(1-s,a,b) = \Gamma_\pi(s)\Big(e^{\pi is/2}L(s,b,-a) + e^{-\pi is/2}e^{2\pi ia}L(s,1-b,a)\Big). \tag{5.1}$$

Changing parameters $a\to 1-a$ and $b\to 1-b$ in (5.1), we obtain

$$e^{2\pi i(1-a)(1-b)}L(1-s,1-a,-b) = \Gamma_\pi(s)\Big(e^{\pi is/2}L(s,1-b,a) + e^{-\pi is/2}e^{2\pi i(1-a)}L(s,b,-a)\Big).$$

Multiplying the both-sides of the formula above by $e^{-2\pi i(1-a)} = e^{2\pi ia}$, we have

$$e^{2\pi i(a-1)b}L(1-s,1-a,-b) = \Gamma_\pi(s)\Big(e^{\pi is/2}e^{2\pi ia}L(s,1-b,a) + e^{-\pi is/2}L(s,b,-a)\Big). \quad (5.2)$$

Moreover, we can easily see that

$$\begin{aligned} Z(s,a,b) &= \sum_{n+a\neq 0} \frac{e^{2\pi i(n+a)b}}{|n+a|^s} = \sum_{n=0}^{\infty} \frac{e^{2\pi i(n+a)b}}{(n+a)^s} + \sum_{n=0}^{\infty} \frac{e^{2\pi i(-n-1+a)b}}{(n+1-a)^s} \\ &= e^{2\pi iab}L(s,a,b) + e^{2\pi i(a-1)b}L(s,1-a,-b), \end{aligned} \quad (5.3)$$

$$\begin{aligned} P(s,a,b) &= \sum_{n+b\neq 0} \frac{e^{-2\pi ina}}{|n+b|^s} = \sum_{n=0}^{\infty} \frac{e^{-2\pi ina}}{(n+b)^s} + \sum_{n=0}^{\infty} \frac{e^{-2\pi i(-n-1)a}}{(n+1-b)^s} \\ &= L(s,b,-a) + e^{2\pi ia}L(s,1-b,a). \end{aligned} \quad (5.4)$$

Hence, from (5.1) + (5.2), we obtain

$$Z(1-s,a,b) = \Gamma_{\cos}(s)P(s,a,b). \quad (5.5)$$

Replacing the variable $s \to 1-s$ in the equality above, we obtain $Z(s,a,b) = \Gamma_{\cos}(1-s)$ $P(1-s,a,b)$. Besides one has $\Gamma_{\cos}(s)\Gamma_{\cos}(1-s) = 1$ by the definition of $\Gamma_{\cos}(s)$ and Euler's reflection formula for the Gamma function. Therefore, we obtain

$$P(1-s,a,b) = \Gamma_{\cos}(s)Z(s,a,b). \quad (5.6)$$

The equations (5.5) and (5.6) imply Riemann's functional equation of (2.22).

On the other hand, we have

$$\begin{aligned} Y(s,a,b) &= \sum_{n+a\neq 0} \frac{\operatorname{sgn}(n+a)e^{2\pi i(n+a)b}}{|n+a|^s} = \sum_{n=0}^{\infty} \frac{e^{2\pi i(n+a)b}}{(n+a)^s} - \sum_{n=0}^{\infty} \frac{e^{2\pi i(-n-1+a)b}}{(n+1-a)^s} \\ &= e^{2\pi iab}L(s,a,b) - e^{2\pi i(a-1)b}L(s,1-a,-b), \end{aligned} \quad (5.7)$$

$$\begin{aligned} O(s,a,b) &= i\sum_{n+b\neq 0} \frac{\operatorname{sgn}(n+b)e^{-2\pi ina}}{|n+b|^s} = i\sum_{n=0}^{\infty} \frac{e^{-2\pi ina}}{(n+b)^s} - i\sum_{n=0}^{\infty} \frac{e^{-2\pi i(-n-1)a}}{(n+1-b)^s} \\ &= iL(s,b,-a) - ie^{2\pi ia}L(s,1-b,a). \end{aligned} \quad (5.8)$$

Thus, by (5.1) − (5.2), we obtain

$$Y(1-s,a,b) = \Gamma_{\sin}(s)O(s,a,b). \quad (5.9)$$

Furthermore, we have

$$O(1-s,a,b) = \Gamma_{\sin}(s)Y(s,a,b) \quad (5.10)$$

from the equations (5.9) and $\Gamma_{\sin}(s)\Gamma_{\sin}(1-s) = 1$ which is proved by Euler's reflection formula. The functional equations (5.9) and (5.10) imply the second equation in (2.22). □

Obviously, we have Theorem 2.4 (1), (2) and (3) by the definitions of $Q(s,a,b)$ and $X(s,a,b)$. The Fourier expansions in (2.23) are easily proved by the functional equations (5.5) and (5.9), namely,

$$Z(s,a,b) = \Gamma_{\cos}(1-s)P(1-s,a,b) \quad \text{and} \quad Y(s,a,b) = \Gamma_{\sin}(1-s)O(1-s,a,b).$$

The functional equations in Theorem 2.4 are easily shown by the integral representations in (2.18). Thus, we only have to prove Theorem 2.4 (0).

Proof of (2.18) for $Q(s,a,b)$. Assume $0 < a, b < 1$ and $\Re(s) > 1$. Then we have

$$2\int_0^\infty u^{s-1}G_Z(u,a,b)du =$$

$$2e^{2\pi iab}\sum_{n=0}^{\infty} e^{2\pi inb}\int_0^\infty u^{s-1}e^{-\pi u^2(n+a)^2}du + 2e^{2\pi i(a-1)b}\sum_{n=0}^{\infty} e^{-2\pi inb}\int_0^\infty u^{s-1}e^{-\pi u^2(n+1-a)^2}du.$$

The first infinite series can be expressed as

$$\sum_{n=0}^{\infty} e^{2\pi inb} \int_0^\infty e^{-v}\left(\frac{v/\pi}{(n+a)^2}\right)^{s/2-1} \frac{dv/\pi}{(n+a)^2} = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\sum_{n=0}^{\infty}\frac{e^{2\pi inb}}{(n+a)^s}. \tag{5.11}$$

Hence, by using (5.3) and (5.11), we obtain

$$2\int_0^\infty u^{s-1}G_Z(u,a,b)du = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Z(s,a,b). \tag{5.12}$$

Similarly, when $\Re(s) > 1$, one has

$$2\int_0^\infty u^{s-1}G_P(u,a,b)du =$$
$$2\sum_{n=0}^{\infty} e^{-2\pi ina}\int_0^\infty u^{s-1}e^{-\pi u^2(n+b)^2}du + 2e^{2\pi ia}\sum_{n=0}^{\infty}\int_0^\infty u^{s-1}e^{2\pi ina-\pi u^2(n+1-b)^2}.$$

Thus, from (5.4) and (5.11), we have

$$2\int_0^\infty u^{s-1}G_P(u,a,b)du = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)P(s,a,b). \tag{5.13}$$

Therefore, when $\Re(s) > 1$, it holds that

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Q(s,a,b) = \int_0^\infty u^{s-1}G_Q(s,a,b)du. \tag{5.14}$$

Hence, from the first modular relation in (2.15), we obtain

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Q(s,a,b) = \int_1^\infty u^{s-1}G_Q(u,a,b)du + \int_0^1 u^{s-1}G_Q(u,a,b)du$$
$$= \int_1^\infty u^{s-1}G_Q(u,a,b)du + \int_1^\infty v^{1-s}G_Q(v^{-1},a,b)\frac{dv}{v^2} = \int_1^\infty \big(u^s + u^{1-s}\big)G_Q(u,a,b)\frac{du}{u}.$$

The last integral converges absolutely for all $s \in \mathbb{C}$, and so the formula holds, by analytic continuation, for all $s \in \mathbb{C}$. Thus we have the first equality in (2.18) for all $0 < a < 1$ and $s \in \mathbb{C}$. From the periodicities of $G_Q(u,a,b)$ and $Q(s,a,b)$, the first equation in (2.18) holds for all $a \in \mathbb{R}\setminus\mathbb{Z}$. Moreover, we have the first equation in (2.18) for all $b \in \mathbb{R}\setminus\mathbb{Z}$ by the quasi-periodicities of $G_Q(u,a,b)$ and $Q(s,a,b)$. ☐

Proof of (2.18) for $X(s,a,b)$. Let us suppose $a, b \in \mathbb{R}\setminus\mathbb{Z}$. By (5.12) and (5.13),

$$2\frac{\partial}{\partial b}\int_0^\infty u^{s-1}G_Z(u,a,b)du = \frac{\partial}{\partial b}\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)Z(s,a,b),$$
$$2\frac{\partial}{\partial b}\int_0^\infty u^{s-1}G_P(u,a,b)du = \frac{\partial}{\partial b}\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)P(s,a,b)$$

when $\Re(s) > 1$. From (3.5), one has

$$\frac{\partial}{\partial b}\int_0^\infty u^{s-1}G_Z(u,a,b)du = 2\pi i\int_0^\infty u^{s-1}G_Y(u,a.b)du,$$
$$\frac{\partial}{\partial b}\int_0^\infty u^{s-1}G_P(u,a,b)du = 2\pi i\int_0^\infty u^{s+1}G_O(u,a)du$$

if $\Re(s) > 1$. Clearly, we can see that

$$\frac{\partial}{\partial b}Z(s,a,b) = \sum_{n+a\neq 0}\frac{\partial}{\partial b}\frac{e^{2\pi i(n+a)b}}{|n+a|^s} = 2\pi i\sum_{n+a\neq 0}\frac{\mathrm{sgn}(n+a)e^{2\pi i(n+a)b}}{|n+a|^{s-1}} = 2\pi i Y(s,a,b),$$

$$\frac{\partial}{\partial b}P(s,a,b) = \sum_{n+b\neq 0}\frac{\partial}{\partial b}\frac{e^{-2\pi ina}}{|n+b|^s} = -s\sum_{n+b\neq 0}\frac{\mathrm{sgn}(n+b)e^{-2\pi ina}}{|n+b|^{s+1}} = -\frac{s}{i}O(s,a,b)$$

when $\Re(s) > 2$. By the formulas above, we have

$$\frac{\partial}{\partial b}\Gamma\Big(\frac{s}{2}\Big)Z(s,a,b) = 2\pi i\Gamma\Big(\frac{s}{2}\Big)Y(s-1,a,b),$$
$$\frac{\partial}{\partial b}\Gamma\Big(\frac{s}{2}\Big)P(s,a,b) = is\Gamma\Big(\frac{s}{2}\Big)O(s+1,a,b) = 2i\Gamma\Big(\frac{s+2}{2}\Big)O(s+1,a,b)$$

for all $s \in \mathbb{C}$. Therefore, if $\Re(s) > 1$, we obtain

$$\int_0^\infty u^s G_Y(u,a,b)du = \pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)Y(s,a,b),$$
$$\int_0^\infty u^s G_O(u,a,b)du = \pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)O(s,a,b).$$

The equations above imply

$$\pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)X(s,a,b) = \int_0^\infty u^s G_X(u,a,b)du.$$

By applying the second modular relation in (2.15), we have

$$\pi^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big)X(s,a,b) = \int_1^\infty u^s G_X(u,a,b)du + \int_0^1 u^s G_X(u,a,b)du$$
$$= \int_1^\infty u^s G_X(u,a,b)du + \int_1^\infty v^{-s} G_X(v^{-1},a,b)\frac{dv}{v^2} = \int_1^\infty \big(u^s + u^{1-s}\big)G_X(u,a,b)du.$$

Obviously, the last integral converges absolutely for all $s \in \mathbb{C}$. Thus formula above holds for all $s \in \mathbb{C}$ by analytic continuation. Hence we obtain the second equality in (2.18) for all $a, b \in \mathbb{R} \setminus \mathbb{Z}$ and $s \in \mathbb{C}$. $\Box$

Acknowledgments. The author was partially supported by JSPS, grant no. 22K03276.

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