Gravity-Electromagnetic Coupling for g-2

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(Dated: May 15, 2024)

Abstract

We investigate the effects of gravity on the electron anomalous magnetic moment, $a_e = (g-2)/2$, within the framework of Quantum Electrodynamics (QED). The application of the Dirac-based gravitational theory reveals significant electromagnetic-gravitational interactions, leading to second-order relativistic effects on a_e estimated at 0.7×10^{-9} in Earth-based experiments. This estimation falls within the observable range, indicating the necessity to consider gravitational effects in high-precision measurements and suggesting novel experimental avenues in gravitational theory.

Keywords

anomalous magnetic moment, g-2, electromagnetic-gravitational interactions, Dirac equation, gravitational theory, Quantum Electrodynamics, QED, Foldy-Wouthuysen transformation

INTRODUCTION

The consideration of gravitational effects on the electron anomalous magnetic moment $a_e = (g-2)/2$ remains an underexplored area of research. In 1947, the anomaly of the electron magnetic moment was discovered by Kusch et al. at a value of $g = 2.00119 \pm 0.00005$, contrary to the strict prediction of the electron's g-factor as 2 according to the Dirac equation [1]. In 2008, the Harvard group achieved a measurement of a_e with 12 digits of precision, and later enhanced this measurement's precision by a factor of 2.2, setting the most accurate determination of a_e to date [2, 3].

$$a_e = 0.00115965218059(13)[0.18\ \text{ppt}] \tag{1}$$

While these experimental achievements, the theoretical investigation of a_e has primarily been explained through QED perturbative calculations, notably pioneered by Schwinger in the past and further refined by Kinoshita et al [4, 5]. However, the effects of gravity on the electron magnetic moment have not been considered, possibly due to reasons such as its minimal impact or the challenges associated with establishing a coupling between gravitational and electromagnetic fields within conventional gravitational theories [6–8].

On the other hand, recent advancements in describing gravity prompt a reconsideration of gravitational influences. In particular, the Dirac-based gravitational theory by Fujita is fundamentally basic yet novel one [9]. This theory originates from the fundamental concept of including gravitational interactions into the Dirac equation, as represented by the following equation:

$$\left[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \left(m - \frac{GmM}{r}\right)\beta\right] = E\psi \tag{2}$$

When the theory is expressed within the QED Lagrangian density framework, it takes the following form:

$$\begin{aligned}\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - eA^\mu\overline{\psi}\gamma_\mu\psi - m(1+g\mathcal{G})\overline{\psi}\psi \\ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\mathcal{G}\partial^\mu\mathcal{G}\end{aligned} \tag{3}$$

This formulation ensures the preservation of the local gauge symmetry, while also describing gravitational interactions in a manner that should include conventional gravitational potential $V(r) = -\frac{GmM}{r}$. Additionally, it has demonstrated consistency in several notable experiments, including the equivalence between inertial and gravitational masses together with the leap second problem [10–12].

When considering the relativistic effects of the Lagrangian density (3), both the gravitational field $\mathcal{G}$ and the electromagnetic field $\boldsymbol{A}$ can interact with each other. This is because, while $\boldsymbol{A}$ represents a vector field, $\mathcal{G}$ is characterized as a scalar field including β. In fact, we have performed calculations for nonrelativistic approximation based on (3). Specifically, utilizing the Hamiltonian derived from this expression and implementing the Foldy-Wouthuysen transformation [13], we have obtained the following term as one of the second-order relativistic effects:

$$H'_{GE} = \frac{1}{mc^2}\left(\frac{G_0Mm}{R}\right)\left(\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right) \tag{4}$$

Eq.(4) implies that the gravitational field influences the g-factor in the Zeeman effect. Furthermore, we should note that these effects should amount to g-2 measurements at an order of 10^{-9}, when evaluating the effects of electrons on Earth.

$$\frac{1}{c^2}\left(\frac{G_0 M}{R}\right) = 0.7 \times 10^{-9} \tag{5}$$

Considering the existing experimental precision of the electron magnetic moment to be 10^{-12}, this effect is within a sufficiently observable range[3].

In summary, this paper presents an investigation into the introduction of the gravitational field within the framework of QED. Specifically, we reveal the emergence of a second-order relativistic effect representing the interaction between the electromagnetic and gravitational fields. Importantly, our findings demonstrate that this interaction significantly influences the electron anomalous magnetic moment a_e at an observable range.

QED LAGRANGIAN DENSITY WITH GRAVITY

The Dirac-based gravitational theory, formulated by Fujita, is built upon the experimentally reliable Dirac theory [9]. Its primary aim is to replicate the experimentally observed gravitational potential $V(r) = -\frac{GmM}{r}$. Notably, this theory characterizes the gravitational field while respecting the local gauge invariance, a requirement typically satisfied by QED. Specifically, the terms concerning gravitational interaction are represented as follows:

$$\mathcal{L}_g = -mg\mathcal{G}\overline{\psi}\psi + \frac{1}{2}\partial_\mu \mathcal{G}\partial^\mu \mathcal{G} \tag{6}$$

Here, g represents the gravitational constant, while $\mathcal{G}$ denotes a gravitational scalar field. The gravitational terms are determined to fulfill the sole guiding principle of reproducing the characteristic features of gravity, ensuring consistency with two conditions: the reproduction of the gravitational potential $V(r) = -\frac{GmM}{r}$ and the equivalence of gravitational and inertial masses. The first condition necessitates the description of the gravitational field as a scalar field. If it were described as a vector field with directional properties like the gauge field, it would invariably fail to satisfy the gravitational condition of being an attractive force. Indeed, we can verify that $\mathcal{G}$ can reproduce the conventional gravitational potential by solving the Euler-Lagrange equation derived from (6), while taking into account $G = \frac{g^2}{4\pi}$.

$$mg\mathcal{G} = -\frac{GmM}{r} \tag{7}$$

Eq.(7) indicates that the equivalence of inertial mass m_i and gravitational mass m_g must hold true.

In the scenario described by (6), higher-order relativistic effects allow for interactions between the electromagnetic and gravitational fields. To investigate relativistic effects, it is necessary to reformulate eq.(6) into the form of the Hamiltonian using the energy-momentum tensor, which can be expressed as follows:

$$H = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A}) + \left(m - \frac{GmM}{r}\right)\beta - \frac{Ze^2}{r} \tag{8}$$

From a non-relativistic approximation of this Hamiltonian, we can obtain a coupling effect between the electromagnetic and gravitational fields.

FOLDY-WOUTHUYSEN TRANSFORMATION

the outline of Foldy-Wouthuysen transformation

Foldy-Wouthuysen transformation(Foldy transformation) is a unitary transformation which decouples the Dirac equation into two two-component equations: one reduces to the Pauli description in the non-relativistic limit; the other describes the negative-energy states [13].

Specifically, first decompose the Dirac Hamiltonian into two components: the odd term $\mathcal{O} = \boldsymbol{\alpha} \cdot \boldsymbol{p}$ and the remaining terms, even term $\mathcal{E}$, which includes potential interactions such as gravitational or Coulomb potentials.

$$H = \beta m + \mathcal{O} + \mathcal{E} \tag{9}$$

The aim is to eliminate the odd terms through a unitary transformation $\psi' = e^{iS}\psi$, where S is expanded in powers of $\frac{1}{m}$, yielding:

$$H' = \beta m + \mathcal{E} + \frac{1}{m}A + \frac{1}{m^2}B + \cdots \tag{10}$$

In concrete, when this unitary transformation is applied, the Hamiltonian can be explicitly expressed in the following form:

$$\begin{aligned} H' =& H + i[S,H] + \frac{i^2}{2!}[S,[S,H]] + \frac{i^3}{3!}[S,[S,[S,H]]] \\ &- \dot{S} - i[S,\dot{S}] + \frac{1}{2}[S,[S,\dot{S}]] + \cdots \end{aligned} \tag{11}$$

Choosing $S = -\frac{i\beta\mathcal{O}}{2m}$, we construct terms to eliminate $\mathcal{O}$ in $i[S, H]$, yielding the following expression for H':

$$H' = \beta\left(m + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m}\right) + \mathcal{E} - \frac{1}{8m^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - \frac{i}{4m^2}[\mathcal{O}, \dot{\mathcal{O}}] + \frac{\beta}{2m}[\mathcal{O}, \mathcal{E}] - \frac{\mathcal{O}^3}{3m^2} + \frac{i\beta\dot{\mathcal{O}}}{2m} \tag{12}$$

Here, we utilize the relationship $\beta\mathcal{O} = -\mathcal{O}\beta$ and the expansion is carried out to the order of $\frac{1}{mc^3}$ and $\frac{1}{m^2c^2}$. By iterating this procedure, the off-diagonal elements naturally vanish, leading to a non-relativistic approximation. While the intricate calculations are omitted here, the result utilized in this study is the third-order Foldy transformation, specified as below [13]:

$$H''' = \beta\left(m + \frac{\mathcal{O}^2}{2m} - \frac{\mathcal{O}^4}{8m}\right) + \mathcal{E} - \frac{1}{8m^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - \frac{i}{4m^2}[\mathcal{O}, \dot{\mathcal{O}}] \tag{13}$$

A coupling between the gravitational and electromagnetic fields can be derived by applying eq.(13) to the Hamiltonian described as (8).

The Foldy transformation of the QED Lagrangian density including the gravitational field

When focusing on deriving the term that couples the electromagnetic potential $\boldsymbol{A}$ with the gravitational field $\mathcal{G}$, it suffices to consider the Foldy transformation in the cases where:

$$\mathcal{O} = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A}) \qquad \mathcal{E} = -\frac{GmM}{r}\beta \tag{14}$$

Substituting eq.(14) into eq.(13) yields the non-relativistic approximation of the total Hamiltonian up to the second relativistic order. Notably, the coupling between $\boldsymbol{A}$ and $\mathcal{G}$ originates specifically from the term $\frac{1}{8m^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]]$, yielding the following result:

$$\begin{aligned} \frac{1}{8m^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] = \frac{1}{8m^2}[&-GmM\delta^3(\boldsymbol{r}) - 4\frac{GmM}{r}\boldsymbol{p}^2 \\ &- 4e^2\frac{GmM}{r}\boldsymbol{A}^2 + 8e\frac{GmM}{r}(\boldsymbol{A}\cdot\boldsymbol{p}) - 4ei\frac{GmM}{r^3}(\boldsymbol{A}\cdot\boldsymbol{r}) \\ &+ 2i\frac{GmM}{r^3}(\boldsymbol{r}\cdot\boldsymbol{p}) + 4ie\frac{GmM}{r}\boldsymbol{\sigma}\cdot(\boldsymbol{p}\times\boldsymbol{A}) \\ &+ 4ie\frac{GmM}{r}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{p}) - 2i\frac{GmM}{r^3}\boldsymbol{\sigma}\cdot(\boldsymbol{r}\times\boldsymbol{p})] \end{aligned} \tag{15}$$

Here, we make use of the relations $(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$, along with the properties $\beta\mathcal{O} = -\mathcal{O}\beta$.

While there are several terms that couple $\boldsymbol{A}$ with $\mathcal{G}$ in eq.(15), the seventh term specifically contributes to the electron magnetic moment:

$$H'_{GE} = \frac{1}{8m^2}4ie\frac{GmM}{r}\boldsymbol{\sigma}\cdot(\boldsymbol{p}\times\boldsymbol{A}) \tag{16}$$

Replacing $\boldsymbol{p}\times\boldsymbol{A}$ with $-i\boldsymbol{B}$ in natural units simplifies the expression, making the interpretation more straightforward:

$$H'_{GE} = \frac{1}{mc^2}\left(\frac{GMm}{r}\right)\left(\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\right) \tag{17}$$

Recalling $\mu_B = \frac{e\hbar}{2m}$, we understand that this term affects the value of the electron magnetic moment.

While equation (14) generates coupling terms between $\boldsymbol{A}$ and $\mathcal{G}$, it is not always guaranteed that terms originating from $\mathcal{O}$ and $\mathcal{E}$ necessarily emerge from the expression $\frac{1}{8m^2}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]]$. For instance, in the case of the Coulomb field described by $\mathcal{E} = -\frac{Ze^2}{r}$, there is no coupling between the electromagnetic potential $\boldsymbol{A}$ and the Coulomb field itself. This distinction arises from whether the Dirac matrix β is included. In case of the gravitational field, β is necessary to satisfy the gravitational conditions outlined in eq.(7), and it results in a reversal of signs when commutating the term $\boldsymbol{\alpha}\cdot\boldsymbol{A}$ within $\mathcal{O}$. However, the Coulomb field does not include β, resulting in no reversal of signs and thus no coupling between $\boldsymbol{A}$ and $\mathcal{G}$. The difference leads to the emergence of terms significantly affecting the magnetic moment or not.

EVALUATION OF GRAVITY-ELECTROMAGNETIC COUPLING TERM

When evaluating the effect of the electron's g-factor, as deduced from eq.(17), we need to pay attention to the specific term identified as C, outlined as follows:

$$C = \frac{1}{c^2}\left(\frac{G_0M}{R}\right) \tag{18}$$

Here, G_0 represents the universal gravitational constant and R denotes the relative distance between the interacting bodies.

At first, consider the simplest but most important case of this effect on Earth. Utilizing Earth's parameters—where the Earth's mass is denoted as M, the electron's mass as m, and the polar radius of Earth as R—we evaluate eq.(18). This yields:

$$C = \frac{1}{c^2}\left(\frac{G_0M}{R}\right) = 0.7\times10^{-9} \tag{19}$$

Here, the Earth's radius is taken as $R = 6356.752$ km [14]. Considering the precision of the electron anomalous magnetic moment a_e, which has been observed with an accuracy exceeding 12 digits, it can be conclusively stated that the effects are within an observable range [3].

Additionally, for theoretical comparison with observations on Earth, measurements performed within artificial satellites could be considered. For instance, in the case of geostationary satellites such as Himawari, which orbit the Earth once every day, we can estimate the orbit radius to be approximately $R = 42000$ km from the Newton's equation. If measurements were performed in this environment, C would take the following value:

$$C = \frac{1}{c^2}\left(\frac{G_0 M}{R}\right) = 0.1 \times 10^{-9} \tag{20}$$

This indicates that measurements in satellite environments could yield differences approximately 7 times smaller than those on Earth's surface, offering a unique perspective for assessing the gravitational-electromagnetic coupling effect.

In summary, the evaluated values for the gravitational-electromagnetic coupling term C are within the precision of 12 decimal places, signifying their observability and their impact on the anomalous electron magnetic moment.

CONCLUSIONS

In this paper, we investigated the gravitational effects within the framework of Quantum Electrodynamics (QED), with a specific focus on the interaction between gravitational and electromagnetic field. We initiated with a QED Lagrangian density that includes gravitational interactions while ensuring gauge invariance and the conditions that gravity must satisfy.

Through the application of the Foldy transformation, we revealed the emergence of coupling terms between the gravitational and electromagnetic fields as the second-order relativistic effects. Notably, a significant discovery within our findings is a term that impacts the electron magnetic moment, specifically denoted as (17).

Moreover, we evaluated the effect of the coupling term on the g-factor, revealing effects ranging from 10^{-9} to 10^{-10} in Earth-based or static satellite-based experiments. This indicates that it influences the anomalous electron magnetic moment a_e within an observable range. Furthermore, gravity-electromagnetic coupling term is independent of the mass of the particle, indicating that the effect is not limited to electron magnetic moments but also extends to other particles like muons.

In conclusion, the emergence of gravity-electromagnetic coupling encourages a reconsideration of anomalous magnetic moments from both theoretical and experimental perspectives. Moreover, it may opening up new avenues for theoretical and experimental investigations in gravitational theory.

ACKNOWLEDGMENTS

The authors would like to express their gratitude to T.Fujita, K.Hanzawa and N.Ikeya for their valuable discussions and insightful contributions to this work.

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