Proof that Collatz conjecture is positive using the classification in binary, the general term of progression of differences and graph theory

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Abstract

This paper presents a new proof of Collatz conjecture using the classification in binary, a multiple of 3, the general term of progression of differences and graph theory. When Collatz process is done, we focus on numbers. Many sequences of numbers are generated. It is the progression of differences. The general term of progression of differences are computed. Then, the proof of contradiction, a multiple of 3, the general term of progression of differences and the Inverse-Collatz process are used to prove that all positive odd other than one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process. Using the classification in binary, we focus on the number of digits. We calculate the expected values of digit (multiply 3 and add 1) (A) and (divide by 2) (B). Comparing the expected values of A and B, we find that there are unequal (B is greater than or equal to A). Thus, Collatz process does not diverge to positive infinity and eventually reaches one digit in binary. Since one digit obtained from Collatz process in binary is equal to 1 in decimal, number of times that the Collatz process reaches 1 is limited.

In graph theory, Collatz process is a directed graph and there is no closed path in a directed graph from the Collatz process. It means that all positive odd other than one do not enter an infinite loop. Therefore, we clarify that Collatz conjecture is positive.

keywords: Collatz Conjecture, 3x+1, progression of differences, expected value, digit, binary, classification in binary, a multiple of 3, Inverse-Collatz process, graph theory

1 Introduction

A problem by Lothar Collatz in 1937, also called $3x+1$ mapping, $3x+1$ problem, kakutani's problem, syrucuse algorithm and Ulam's problem (Lagrarias 1985)

[1]. The problem is simple. Pick a number, any positive integer. If it is odd, multiply it by 3 and add 1. If it is even, divide it by 2. Now you have a new number. Apply the same rules to the new number. The conjecture is about what happens as you keep repeating the process. Over the years, many mathematicians are trying to the proof of this conjecture. Terence Tao posted the proof of this conjecture for "almost all numbers" in 2022 [2]. But there is no proof that Collatz conjecture is positive or negative. Now we report the new proof using the binary, the general term of progression of differences and graph theory, to clarify whether Collatz conjecture is positive or negative. If selected number is even then it reaches to odd by Collatz process. So, we take positive odd mainly to this proof.

2 First proof

The proof of contradiction and the general term of progression of differences are used to prove that all positive odd other than one do not enter an infinite loop (e.g.$1 \to 3 \to 4 \to 1$) by one cycle of Collatz process. For 2 or more cycles of Collatz process, proof is shown later.

2.1 Regularities in the numbers

Regularities in the numbers are derived from Collatz process. If we choose 1, multiply 1 by 3, add 1 and divide it by 2 (Collatz process), the choosed one is converged to 1, the same for 5, 21, 85, 341, $\cdots$. If we choose 3 and calculate it using the Collatz process, we get the results that 3 is converged to 5, the same for 13, 53, 213, 853, $\cdots$. It is the progression of differences. Table 1 and 2 show the results of Collatz process. The odd column is described odd from 1 to 15.

Table 1 Results of Collatz process from first to third term of sequence

Positive integer shown as k	Odd represented by 2k+1	Convergent type	First term of sequence	Second term of sequence	Third term of sequence
0	1	1	1	5	21
1	3	5	3	13	53
2	5	1	5	21	85
3	7	11	7	29	117
4	9	7	9	37	149
5	11	17	11	45	181
6	13	5	13	53	213
7	15	23	15	61	245
Moreover	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 2 Results of Collatz process from fourth to fifth term of sequence and moreover

Positive integer shown as k	Odd represented by 2k+1	Convergent type	Fourth term of sequence	Fifth term of sequence	Moreover
0	1	1	85	341	$\cdots$
1	3	5	213	853	$\cdots$
2	5	1	341	1365	$\cdots$
3	7	11	469	1877	$\cdots$
4	9	7	597	2389	$\cdots$
5	11	17	725	2901	$\cdots$
6	13	5	853	3413	$\cdots$
7	15	23	981	3925	$\cdots$
Moreover	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

2.2 The general term of progression of differences

We calculate the general term of progression of differences using the numbers from Collatz process. Odd numbers of 3, 13, 53, 213, 853, $\cdots$ from line second in table 1 and 2 are converged to 5 by Collatz process. This is the progression of differences, $a_n = a_1 + 10\sum_{k=1}^{n-1} 4^{k-1}$. The first term of sequence is 3 and common ratio is 4. General term is shown below.

$$a_n = 3 + \frac{10(4^{n-1} - 1)}{3} \tag{1}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

Similarly odd numbers of 1, 5, 21, 85, 341, $\cdots$ and 7, 29, 117, 469, 1877, $\cdots$ from line first and fourth in table 1 and 2 are converged to 1 and 11 by Collatz process. General terms of sequence are shown below.

Odd numbers of 1, 5, 21, 85, 341, $\cdots$ from line first in table 1 and 2

$$a_n = 1 + \frac{4(4^{n-1} - 1)}{3} \tag{2}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

Odd numbers of 7, 29, 117, 469, 1877, $\cdots$ from line fourth in table 1 and 2

$$a_n = 7 + \frac{22(4^{n-1} - 1)}{3} \tag{3}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

About the all positive odd, we can represent as 2k+1 (k is a positive integer including zero). Then, the first term of sequence is 2k+1 and common retio is 4. The general term of progression of differences is shown below.

$$a_n = (2k+1) + \frac{2(3k+2)(4^{n-1}-1)}{3} \tag{4}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

Prove that equation No.4 holds true for k (k is a positive integer including zero).

i) When k=0, equation No.4 is shown below.

$$a_n = 1 + \frac{4(4^{n-1}-1)}{3} \tag{5}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

ii) When k=1, equation No.4 is shown below.

$$a_n = 3 + \frac{10(4^{n-1}-1)}{3} \tag{6}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

iii) When equation No.6 - equation No.5, a result is shown below.

$$a_n = 2 \times 4^{n-1} \tag{7}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

iv) Assume k = g (g is a positive integer including zero) holds true. The equation No.4 becomes the following equation.

$$a_n = (2g+1) + \frac{2(3g+2)(4^{n-1}-1)}{3} \tag{8}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

v) When k = g + 1, equation No.4 becomes the following equation.

$$a_n = (2g+3) + \frac{2(3g+5)(4^{n-1}-1)}{3} \tag{9}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

vi) When equation No.9 - equation No.8, a result is shown below.

$$a_n = 2 \times 4^{n-1} \tag{10}$$

(Here n represents a natural number more than 2 and n holds true even if 1.)

From iii) and vi), $(No.6 - No.5) = (No.9 - No.8)$. When k=g and k=g+1, we know that the equation No.4 holds true. We are able to state that the equation No.4 holds true for all positive integer including zero by mathematical induction method. Since we describe the all positive odd as 2k+1 (k is a positive integer including zero), No.4 holds true for all positive odd.

Let multiply the No.4 by 3 and add 1, the No.4 will become the No.11. No.11 is shown below.

$$a_n = 2(3k + 2) \times 4^{n-1} \tag{11}$$

(Here n represents a natural number more than 2 and n holds true even if 1. k is a positive integer including zero.)

No.11 is even and just before divide by 2.

2.3 Proposition

Prove that all positive odd other than 1 do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by one cycle of Collatz process.

2.4 Proof

Assume, for the sake of contradiction, that 2 or more positive odd numbers including 1 enter infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process. Note that there exists the number returning to original odd same as 1. It is unknown but we can be written as 2m+1, where m is the positive integer including zero.

As described the subsection 2.2, No.4 is proved to be true for all positive odd, then No.11 just before dividing by 2 is true for all positive odd. In accordance with this, No.11 has a number returning to original odd. In this sense, we can get the original odd dividing No.11 by 2 to enter an infinite loop. Then, this original odd is the same as 2m+1.

Let divide No.11 by 2 repeatedly until appearance of odd number. An odd obtained by dividing 2 is the equal to original odd number. If k is odd, then No.11 becomes 3k+2(this is odd). So, we show the equation 3k+2 = 2m+1(No.31). Even though there are many solutions in No.31, there is no solution that k and m are the same in positive integer including zero.

If k is even, then k is classified two types. For the first type, k is zero and $odd \times 2^n$, when n represents a natural number and $n \geq 2$. For the second type, k is $odd \times 2^n$, when n=1. In the first type, No.11 becomes $\frac{3}{2}k + 1$(this is odd). So, we show that the equation $\frac{3}{2}k + 1 = 2m + 1$(No.32). Even though there are many solutions in No.32, only zero is the solution that k and m are the same in positive integer including zero. In the second type, No.11 becomes $\frac{3}{4}k + \frac{1}{2}$(this is odd). So, we show the equation $\frac{3}{4}k + \frac{1}{2} = 2m + 1$(No.33). Even though there are many solutions in No.33, there is no solution that k and m are the same in positive integer including zero.

From the above, only zero is the solution that k and m are the same in positive integer including zero. So, there exists the only one infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process.

This is the contradiction.

Only 1 enters an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process, but all positive odd other than one do not enter an infinite loop. Therefore, it must be true that all positive odd except one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by one cycle of Collatz process. For 2 or more cycles of Collatz process, proof is shown next.

3 Second proof

A multiple of 3, a classification of odd in binary, the classified numbers from Collatz process, and the Inverse-Collatz process are used to prove that all positive odd except 1 do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by 2 or more cycles of Collatz process.

3.1 A multiple of three

The number after the Collatz process does not contain multiples of 3. The Collatz process is an operation in which, when a number is chosen, if it is odd, it is multiplied by 3 and then add 1 to make it an even number, and then divided by 2 until it becomes an odd number. If an odd number is expressed as 2n+1 (n is a positive integer containing 0), then when multiplied by 3, it becomes 6n+3 (n is a positive integer containing 0), and when 1 is added to it, it becomes 6n+4 (n is a positive integer containing 0). Since 6n+4=2(3n+2), we know that we can determine whether 3n+2 (n is a positive integer containing 0) is a multiple of 3 or not. Since 3n+2 (n is a positive integer including 0) is a multiple of 3 plus 2 when n is odd, it is not a multiple of 3. When n is even, n is the form $odd \times 2^n$, and 3n+2 takes two types.
1).n takes one, or 2) n takes two or more.
In the case of 2), 3n+2 (n is a positive integer containing 0) can be divided by 2 again, so it takes the form a multiple of 3 plus 1, which is not a multiple of 3. In the case of 1), dividing 3n by 2 results in an odd number, it can be further divided by two again since (odd+1) is an even number. In this case, 3n+2 can be expressed as 3x2x(2m+1)+2 (m is a positive integer including 0), which is divided by two to form 3m+2, which is also a multiple of three plus 2 and not a

multiple of 3. Therefore, it is proved that the number after the Collatz process does not contain multiples of 3.

3.2 Classification using the last three digits in binary

All positive odd numbers before Collatz process are classified into the following four types using the last three digits in binary.

1. Type A: The last three digits in binary are of the type (001). The numbers belonging to this type are 1, 9, 17, 25, 33, $\cdots$. This number sequence can be expressed as 8n+1 (n is a positive integer including 0).

2. Type B: The last three digits in binary are of the type (011). The numbers belonging to this type are 3, 11, 19, 27, 35, $\cdots$. This number sequence can be expressed as 8n+3 (n is a positive integer including 0).

Type C: The last three digits in binary are of the type (101). The numbers belonging to this type are 5, 13, 21, 29, 37, $\cdots$. This number sequence can be expressed as 8n+5 (n is a positive integer including 0).

Type D: The last three digits of this type in binary are of the type (111). The numbers belonging to this type are 7, 15, 23, 31, 39, $\cdots$. The number sequence can be expressed as 8n+7 (n is a positive integer including 0).

3.3 Results of Collatz process in type A,B and D

Type A.

If we do the Collatz process on type A, we get (8n+1)x3+1=24n+4=4(6n+1). Dividing by two, this number becomes 6n+1 (n is a positive integer containing 0). The result of the formula and n are shown in the table below.

Table 3 Results of Collatz process on type A

n	0	1	2	3	4	5	6	7	$\cdots$
6n+1	1	7	13	19	25	31	37	43	$\cdots$

Type B.

If we do Collatz process on type B, we get (8n+3)x3+1=24n+10=2(12n+5). Divided by 2, this number becomes 12n+5 (n is a positive integer containing 0). The result of the formula and n are shown in the table below.

Table 4 Results of Collatz process on type B

n	0	1	2	3	4	5	6	7	$\cdots$
12n+5	5	17	29	41	53	65	77	89	$\cdots$

Type D.

If we do Collatz process on type D, we get (8n+7)x3+1=24n+22=2(12n+11). Divided by 2, this number becomes 12n+11 (n is a positive integer containing 0). The result of the formula and n are shown in the table below.

Table 5 Results of Collatz process on type D

n	0	1	2	3	4	5	6	7	···
12n+11	11	23	35	47	59	71	83	95	···

Summarized type A, B and D.

We have proved that the number after Collatz process does not contain multiples of three. Let assume that the number after Collatz process contain multiples of three. This number becomes 6n+3 (n is a positive integer containing 0). The result of the formula and n is shown in the table below.

Table 6 Numbers after Collatz process contain multiples of three

n	0	1	2	3	4	5	6	7	···
6n+3	3	9	15	21	27	33	39	45	···

The above is summarized in the following table.

Table 7 Summarized numbers after Collatz process contain multiples of three

n	0	1	2	3	4	5	6	7	···
6n+1	1	7	13	19	25	31	37	43	···
12n+5	5	17	29	41	53	65	77	89	···
12n+11	11	23	35	47	59	71	83	95	···
6n+3	3	9	15	21	27	33	39	45	···

From this table 7, it is clear that once n is determined, odd numbers up to 12n+5 at $\frac{n}{2}$ can be arranged in order from 1 when n is even, and odd numbers up to 12n+11 at $\frac{n-1}{2}$ can be arranged in order from 1 when n is odd.

That is, when n is even, the numbers 6n+1, 6n+3 and 12 $\frac{n}{2} + 5$=6n+5 can be arranged in order, and when n is odd, 6n+1, 6n+3 and 12 $\frac{n-1}{2} + 11$=6n+5 can be arranged in order.

3.4 Results of Collatz process in type C

Let consider the case of type C. In binary, type C is a sequence of numbers whose last three digits are (101). The number after the Collatz process is 3n+2 (n is a positive integer including 0). Note that the number may be further divided by 2 depending on the value of n. This number type is characterized by the last letter of (01) in binary. For all positive odd numbers, no matter how many times the (01) is repeated at the end of number in binary, the result of the Collatz process is the same as the odd number before (01) is repeated at the end. For example, 11 (1011 in binary), 45 (101101 in binary), 181 (10110101 in binary), 725 (1011010101 in binary), or 2901 (101101010101 in binary) will all become the same 17 by Collatz processing.

3.5 Classification of numbers in decimal after the Collatz process

To explain the Collatz process briefly, process is to multiply an odd number by 3, add 1, and divide by 2. For this process, picked an odd number is 2k+1 (k is a positive integer including 0), then (2k+1)+ (2k+1)+(2k+1)+1 is obtained by multiplying an odd number by 3, adding 1 and this can be transformed as below.

$$(2k+1)+(2k+1)+(2k+1)+1=(2k+1)+(2k+1)+(2k+2) \tag{12}$$

$$(2k+1)+(2k+1)+(2k+2)=(2k+1)+(2k+1)+(k+1)+(k+1) \tag{13}$$

$$(2k+1)+(2k+1)+(k+1)+(k+1)=(2k+1)+(k+1)+(2k+1)+(k+1) \tag{14}$$

$$(2k+1)+(k+1)+(2k+1)+(k+1)=(3k+2)+(3k+2) \tag{15}$$

This (3k+2) + (3k+2) is divided by 2, so the quotient is (3k+2).

Now, equation 11, which is the equation just before dividing by 2 and this equation can be transformed as below.

$$a_n = 2(3k+2)\times 4^{n-1} = (3k+2)\times 4^{n-1} + (3k+2)\times 4^{n-1} \tag{16}$$

Dividing by 2, the quotient is $(3k+2)\times 4^{n-1}$.
Table 8 shows the relationship between $(3k+2)\times 4^{n-1}$, 2k+1(odd number), k(positive integer containing 0) and n(natural number).

Table 8 Relationship between k(positive integer containing 0), 2k+1(odd number), $(3k+2) \times 4^{n-1}$ and n(natural number)

		n=1	n=2	n=3	n=4	$\cdots$
k	2k+1	$(3k+2) \times 4^0$	$(3k+2) \times 4^1$	$(3k+2) \times 4^2$	$(3k+2) \times 4^3$	$\cdots$
0	1	2	8	32	128	$\cdots$
1	3	5	20	80	320	$\cdots$
2	5	8	32	128	512	$\cdots$
3	7	11	44	176	704	$\cdots$
4	9	14	56	224	896	$\cdots$
5	11	17	68	272	1088	$\cdots$
6	13	20	80	320	1280	$\cdots$
7	15	23	92	368	1472	$\cdots$
8	17	26	104	416	1664	$\cdots$
9	19	29	116	464	1856	$\cdots$
10	21	32	128	512	2048	$\cdots$
11	23	35	140	560	2240	$\cdots$
12	25	38	152	608	2432	$\cdots$
13	27	41	164	656	2624	$\cdots$
14	29	44	176	704	2816	$\cdots$
15	31	47	188	752	3008	$\cdots$
16	33	50	200	800	3200	$\cdots$
17	35	53	212	848	3392	$\cdots$
18	37	56	224	896	3584	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

We can realize the following from table 8.

1. For all odd number described as 2k+1, we can gain the corresponding 3k+2, which is the only number after the Collatz process, and there exists the geometric progression with first term 3k+2 and a common ratio of 4.

2. 3k+2 has two types, For the first type of 3k+2 is $odd \times 2^n$, when n represents a natural number. For the second type of 3k+2 is odd, so the 3k+2 gained by the Collatz process in first type is further divided by 2 until to be the odd.

3. Pick any positive odd number as A, since A is 2k+1, we can compute k and gain the only 3k+2 by the Collatz process. Now that we have gained a 3k+2, we are able to classify the odd A.

For example,

1: Pick the odd number as 3925, 3925 has k=1962 since 2k+1=3925 and 3k+2=5888=23×4^4. Thus, 3925 could be classified as the geometric progression with a first term of 23 due to odd 15 and n=5.

2: Pick the odd number 3927, 2k+1=3927, so k=1963 and 3k+2=5891=5891$\times 4^0$. Thus, 3927 could be classified as the geometric progression with the first term of 5891 due to odd 3927 and n=1.

3: Pick the odd number 9557, 2k+1=9557, so k=4778 and 3k+2=14336=7$\times 2 \times 4^5$. Thus, 9557 could be classified as the geometric progression with first term 7×2 due to odd 9 and n=6. In this case, the odd number obtained by the Collatz process is 7.

From the above, we know that there is the progression of differences $(3k + 2) \times 4^{n-1}$ obtained by Collatz process corresponding to all odd numbers 2k+1 and that odd numbers can be classified. We also showed in the previous section that the numbers are ordered. From these facts, let trace back the odd numbers by computing from 1 in the inverse order of the Collatz process, i.e. The Inverse-Collatz process.

3.6 Inverse-Collatz process

The Collatz process is to multiply an odd number by 3, add 1, and divide by 2 many times until the quotient becomes an odd number. This process is used to move backward from 1. The Inverse-Collatz process is to multiply an odd number by 2 many times (times are once, twice or manytimes) as a power of 2, subtract 1, and divide by 3. If the odd number traced back is a multiple of 3, the next Inverse-Collatz process cannot be performed, and the traversal of odd numbers ends at that point. This can be said to be the end of the Inverse-Collatz process, because no matter what number is multiplied by a multiple of 3, the number is still a multiple of 3, and if 1 is subtracted from the multiple of 3, the number is no longer a multiple of 3, and then the number cannot be divided by 3. Now, the result of multiplying 1 by 2 as a power of 2, subtracting 1 from the resulting number and dividing by 3 is shown in Table 9.

Table 9 The result of multiplying 1 by 2 as a power of 2, subtracting 1, and dividing by 3 (n is a nutural number.)

n	1	2	3	4	5	6	7	8	$\cdots$
1×2^n	2	4	8	16	32	64	128	256	$\cdots$
subtract 1	1	3	7	15	31	63	127	255	$\cdots$
divide by 3		1		5		21		85	$\cdots$

From table 9, the operation that we multiply 1 by 2 as a power of 2, subtract 1 and devide by 3 is completed when n=2, 4, 6, 8, $\cdots$. The completed numbers are shown below as the progression of differences.

$a_1 = 1, a_m = 1 + \frac{4(4^{m-1}-1)}{3}$ (m is a nutural number and m=$\frac{n}{2}$), and this sequence continues indefinitely.
Thus, when performing the Inverse-Collatz process, one is free to choose any number in this sequence, and if the number is not a multiple of 3, it can be used as an odd number for the next Inverse-Collatz process.
From Table 9, choose 5. Table 10 is a table of the Inverse-Collatz process that multiply the chosen odd number 5 by 2 as a power of 2, subtract 1, and divide by 3.

Table 10 The result of multiplying 5 by 2 as a power of 2, subtracting 1, and dividing by 3 (n is a nutural number.)

n	1	2	3	4	5	6	7	$\cdots$
5×2^n	10	20	40	80	160	320	640	$\cdots$
subtract 1	9	19	39	79	159	319	639	$\cdots$
divide by 3	3		13		53		213	$\cdots$

From this table 10, the operation that we multiply 5 by 2 as a power of 2, subtract 1 and devide by 3 is completed when n=1, 3, 5, 7, $\cdots$. The completed numbers are shown below as the progression of differences.
$a_1 = 3, a_m = 3 + \frac{10(4^{m-1}-1)}{3}$ (m is a nutural number and m=$\frac{n+1}{2}$), and this sequence continues indefinitely.
When performing the Inverse-Collatz process, all numbers in this sequence can be freely chosen, and if the number is not a multiple of 3, it can be used as an odd number for the next Inverse-Collatz process. As described above, one can follow the values of the progression of differences from odd 1 and choose numbers one after another, so that one could even reach, for example, odd 27 in this way. Odd 27 is a multiple of 3, so traversing the sequence stops at 27. However, if we choose 109 instead of 27 at the preceding step of odd 41, it is possible to trace the number further back, and we can trace the number indefinitely.

3.7 Proposition

Prove that all positive odd other than 1 do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by 2 or more cycles of Collatz process.

3.8 Proof

All positive odd numbers before Collatz process can be classified into the four types (A,B,C and D) in binary.

We have shown that if all positive odd numbers are classified into four types by the last three digits of the binary system, and if we choose types A, B, and D among them for the Collatz process, once n is determined, the odd numbers after the Collatz process, together with multiples of 3, can be arranged in order

from 1. Then, after the Collatz process for an odd number, there will be only one number in that odd number, and that odd number will not have the same odd number. For the sake of clarity, we have combined multiples of 3, but the result is the same even if a multiple of 3 exists only for the first time and not for second or more subsequent times. Therefore, for odd numbers of types A, B, and D, there is no loop due to the Collatz process (with the exception of one) by 2 or more cycles of Collatz process. Plus, we have shown that C-type odd numbers are odd numbers that take the form of adding a series of (01) at the end in binary to odd A, B, and D types, and that they do not loop infinitely because they only overlap with odd numbers after Collatz's processing to A, B, and D types.
Besides, since the number of powers of 2 in the Inverse-Collatz process described above can be freely chosen, it is not possible to get stuck in a loop that always returns to the original number, and it can be concluded that this does not affect the numerical value traversal.
In other words, there may be a loop but there is also a entrance number and also freely selected exit numbers. So, there is no loop to get stuck in a loop in the Inverse-Collatz process.
The same can be concluded for the Collatz process. A loop that returns to the same number means that one is stuck in the loop, but since there is no loop that one is stuck in when going back through the flow of numbers (Inverse-Collatz process), it can be concluded that the same is true for the original Collatz process.

It is proved that all positive odd other than one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by 2 or more cycles of Collatz process

4 Third proof

It is proved that Collatz process does not diverge to positive infinity and the number of times the Collatz process reaches 1 is limited.

4.1 Multiply in binary

Consider to multiply in binary. 3 in decimal is described 11 in binary. Pick a number, any positive integer, selected number is denoted as (1aaa···aaa) in binary, where a is 0 or 1 regardless of the number of digits (a=0 or a=1). A number multiplied by 11 (3 in decimal) in binary is shown below.

$$(1aaa \cdots aaa) \times 11 =$$

$$(1aaa \cdots aaa) \times 10 + (1aaa \cdots aaa) = (1aaa \cdots aaa0) + (1aaa \cdots aaa) \quad (17)$$

This is increased one digit and added original number in binary.

4.2 Divide in binary

Consider to divide in binary. 2 in decimal is described 10 in binary. Pick a number, any positive even in decimal, selected number is denoted as (1aaa··· aa0) in binary, where a is 0 or 1 regardless of the number of digits (a=0 or a=1). A number divided by 10 (2 in decimal) in binary is shown below.

$$(1aaa\cdots aa0) \div 10 = (1aaa\cdots aa) \tag{18}$$

The one zero elimination, this is decreased one digit in binary.

4.3 Positive odd classification by first 3 digits

Let classify positive odd in binary. Pick a number, any positive odd in decimal, selected number can be classified into four types by judgment of first 3 digits in binary regardless of the number of digits. There exist only four types in binary and every type is shown below.

$(100aaa\cdots aa1), wherea = 0ora = 1, (No.34)$
$(101aaa\cdots aa1), wherea = 0ora = 1, (No.35)$
$(110aaa\cdots aa1), wherea = 0ora = 1, (No.36)$
$(111aaa\cdots aa1), wherea = 0ora = 1, (No.37)$
Positive odd of 3 digits or less in binary will be described later.

4.4 Multiply from No.34 to 37 by 11(3 in decimal) in binary

Let multiply from No.34 to 37 by 11(3 in decimal) in binary. Calculation results are shown below.

$No.34 \times 11$
$(100aaa\cdots aa1) \times 11 =$

$$(100aaa\cdots aa10) + (100aaa\cdots aa1) = (110aaa\cdots a(a+1)1) \tag{19}$$

$No.35 \times 11$
$(101aaa\cdots aa1) \times 11 =$

$$(101aaa\cdots aa10) + (101aaa\cdots aa1) = (111(a+1)aaa\cdots a(a+1)1) \tag{20}$$

$No.36 \times 11$
$(110aaa\cdots aa1) \times 11 =$

$$(110aaa\cdots aa10) + (110aaa\cdots aa1) = (1001aaaa\cdots a(a+1)1) \tag{21}$$

$No.37 \times 11$
$(111aaa\cdots aa1) \times 11 =$

$$(111aaa\cdots aa10) + (111aaa\cdots aa1) = (1010(a+1)aaa\cdots a(a+1)1) \tag{22}$$

Regardless of a, single or double increase in digits is seen from No.19 to No.22. No.34 is multiplied by 11(3 in decimal) and a product becomes No.19. Single increase in digits of No.19 is seen. No.35 is multiplied by 11(3 in decimal) and a product becomes No.20. Single or double increase in digits of No.20 is seen. Single or double increase is related that a=0 or a=1. No.36 and No.37 are multiplied by 11(3 in decimal) and products become No.21 and No.22. Double increase in digits of No.21 and No.22 are seen.

4.5 Calculate the expected value of increased digits

Let calculate the expected value of increased digits, when No.34, 35, 36 and 37 are multiplied by 11(3 in decimal) in binary. Appearance probability P for No.34, 35, 36 and 37 are equal to 0.25. As described the subsection 4.3, there are only four positive odd types in binary. In accordance with this, the each expected value of increased digits is shown below.

No.19 is $0.25 \times 1 = 0.25$

No.20 is $0.25 \times 1 = 0.25$ *or* $0.25 \times 2 = 0.5$ (*at a maximum*)

No.21 is $0.25 \times 2 = 0.5$

No.22 is $0.25 \times 2 = 0.5$

In all positive odd, the expected value of increased digits in binary is 1.75(0.25+0.5+0.5+0.5) at a maximum.

4.6 Consider to add 1 in binary

Consider to add 1 in binary. Pick a number, any positive odd integer, selected number is denoted as $(1aaa \cdots aaa1)$ in binary. If all of a is 1, then single increase in digits by adding 1 is seen. This appearance probability is related to digits number. When the number of digits is 4, probability of appearance is 1/8. Five of digits is same as 1/16. That probability can be neglected, where there are many digits for all positive odd in binary.

4.7 Positive odd classification by last 3 digits

Let classify positive odd in binary again. Pick a number, any positive odd in decimal, selected number can be classified into four types by judgment of last 3 digits in binary. There exist only four types in binary and every type is shown below.

(1aa···aa001), where a=0 or a=1, (No.38)

(1aa···aa011), where a=0 or a=1, (No.39)

(1aa···aa101), where a=0 or a=1, (No.40)

(1aa···aa111), where a=0 or a=1, (No.41)

Positive odd of three digits or less (8 or less in decimal) in binary are proved that each of these numbers do not diverge to positive infinity by Collatz process and reach 1.

4.8 Multiply by 11(3 in decimal) and add 1 in binary

Let multiply No.38, No.39, No.40 and No.41 by 11(3 in decimal) and add 1 in binary. Calculation results are below.

$$(No.38 \times 11) + 1 = (1aa \cdots aa001) \times 11 + 1 =$$

$$(1aa \cdots aa0010) + (1aa \cdots aa001) + 1 =$$

$$(1aa \cdots aaa011) + 1 = (1aa \cdots aaa100) \tag{23}$$

$$(No.39 \times 11) + 1 = (1aa \cdots aa011) \times 11 + 1 =$$

$$(1aa \cdots aa0110) + (1aa \cdots aa011) + 1 =$$

$$(1aa \cdots aa(a+1)001) + 1 = (1aa \cdots aa(a+1)010) \tag{24}$$

$$(No.40 \times 11) + 1 = (1aa \cdots aa101) \times 11 + 1 =$$

$$(1aa \cdots aa1010) + (1aa \cdots aa101) + 1 =$$

$$(1aa \cdots aa(a+1)111) + 1 = (1aa \cdots aa(a+1+1)000) \tag{25}$$

$$(No.41 \times 11) + 1 = (1aa \cdots aa111) \times 11 + 1 =$$

$$(1aa \cdots aa1110) + (1aa \cdots aa111) + 1 =$$

$$(1aa \cdots aa(a+1+1)101) + 1 = (1aa \cdots aa(a+1+1)110) \tag{26}$$

4.9 Divide by 10 (2 in decimal) repeatedly in binary

Let divide No.23, No24, No.25 and No.26 by 10 (2 in decimal) repeatedly in binary until appearance of odd number. Regardless of values of a, the decrease of digits is seen from No.23 to No.26 dividing by 10 (2 in decimal) in binary. Calculation results are shown below.

$$No.23 \div 10(2\ in\ decimal) =$$

$$(1aa \cdots aaa100) \div 10 = \cdots = (1aaa \cdots aa1) \tag{27}$$

The two zeros elimination, this is decreased two digits.

$No.24 \div 10(2\ in\ decimal) =$

$$(1aa\cdots aa(a+1)010) \div 10 = (1aaa\cdots aa(a+1)01) \tag{28}$$

The one zero elimination, this is decreased one digit.

$No.25 \div 10(2\ in\ decimal) =$

$$(1aa\cdots aa(a+1+1)000) \div 10 = \cdots = (1aaa\cdots aa(a+1+1)) \tag{29}$$

The three zeros elimination, this is decreased three digits. Depending on a=0 or a=1, three or more zeros are eliminated. Three or more of digits (3 of digits are at a minimum) are decreased.

$No.26 \div 10(2\ in\ decimal) =$

$$(1aa\cdots aa(a+1+1)110) \div 10 = (1aaa\cdots aa(a+1+1)11) \tag{30}$$

The one zero elimination, this is decreased one digit.

4.10 Calculate the expected value of decreased digits

Let calculate the expected values of decreased digits. Appearance probability P for No.23, No.24, No.25 and No.26 are equal to 0.25. As described the subsection 4.7, there are only four positive odd types in binary. In accordance with this, the each expected value of decreased digits is shown below.

No.27 is $0.25 \times 2 = 0.5$

No.28 is $0.25 \times 1 = 0.25$

No.29 is $0.25 \times 3 = 0.75$ (*at a minimum*)

No.30 is $0.25 \times 1 = 0.25$

In all positive odd, the expected value of decreased digits is 1.75(0.5+0.25+0.75+0.25) at a minimum.

4.11 Proposition

All positive odd do not diverge to positive infinity by Collatz process and the number of times that the Collatz process reaches 1 is limited.

Pick a number, any positive integer. If it is odd, multiply it by 3 and add 1. If it is even, divide it by 2. Now you have a new number. Apply the same rules to the new number. This is the collatz process.

4.12 Proof

About all positive odd numbers, let compare the expected value of decreased digits by dividing 10 (2 in decimal) in binary (shown as B) with the expected value of increased digits by multiplying by 11 (3 in decimal) and adding one in binary (shown as A). Here, the expected value of increased digits by adding 1 can be neglected as described the subsection 4.6. A is equal to B as described the subsection 4.5 and 4.10. But B is at a minimum and A is at a maximum. In according with this, there is inequality (B is greater than or equal to A). Therefore, all positive odd do not diverge to positive infinity by Collatz process.

Although the number of digits may increase by Collatz process, the expected value of decreased digits is greater than or equal to the expected value of increased digits. The expected value is based on the law of large number. Then digits are decreased gradually by Collatz process in binary. Moreover, it is proved at the section 2 and 3 that all positive odd except 1 do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process. Thus, Collatz process does not diverge to positive infinity and eventually reaches the one digit in binary. Since one digit in binary is equal to 1 in decimal, number of times that the Collatz process reaches 1 is limited.

5 Discussion and Conclusion

The Collatz process begins by choosing an odd number, multiplying by 3, adding 1, dividing by 2, and if the quotient is odd, moving on to the next process. When performing a Collatz process, one number always corresponds to one number. Paul. J. Andaloro and Heinz Ebert have shown a graph theroretical approach to the Collatz conjecture in 2002 and 2021 [3][4].
We would like to consider the Collatz process similarly from graph theory [5]. Let put numbers from the Collatz process as vertex and put the Collatz process as edge (incident to vertex), we can say that the Collatz process is a directed graph. D is a directed graph, V is a set of vertex, and A is a set of directed edge.

D = (V, A)

$$V = \{v_1, v_2, v_3, \cdots, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$$

$$A = \{(v_1, v_2), (v_2, v_3), \cdots, (v_{n-2}, v_{n-1}), (v_{n-1}, v_n)\}$$

Since the Collatz process is replaced by a directed graph, the picked number is an endpoint, also called the starting point, whose degree is 1. At every vertex in the middle, indegree=1, outdegree=1 and the degree is 2. Thus, there is an endpoint (terminal point), and its degree is odd. In the Collatz process, one number is always determined for one number, so the degree of the endpoint(terminal point) is 1.

If $v_1 = v_n$ from the above set of vertex and edge, then there exists a closed path in a directed graph. There is no proof to $v_1 \neq v_n$ in the previous graph theroretical approach to the Collatz conjecture(2000 and 2021). In section 4 (Third proof), the proof show that the Collatz process reduces each numerical digit gradually to a single at the end by comparing the numerical variation after the Collatz process with the expected value of the increasing or decreasing digits in the binary system. Therefore, we can say that $v_1 \neq v_n$ and there is no closed path in a directed graph from the Collatz process.

As expressed in section 2 (First proof), 3 (Second proof) and 4 (Third proof), we have proved that all positive odd does not diverge to positive infinity by Collatz process and reach 1 within the limited repeat times. And also, we have proved that all positive odd other than 1 do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$). Therefore, it is concluded that Collatz conjecture is positive. But we cannot calculate the number of repeated times of Collatz process in all positive odd. We consider that this is the remaining problem.

Proof of contradiction using the general term of sequence, graph theory and calculation of the expected value by number classification in binary are very useful in this proof. The focus on digit in binary is a turning point of the proof about Collatz conjecture. Numbers in decimal and digits in binary are related but different concepts. It can be said that each is the different universe. We may be play the positive proof of Collatz conjecture in the "Hodge theater" [6].

6 Declaration

Author has no conflict of interest to declare.

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8 Refference

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