Proof that Collatz conjecture is positive using the classification in binary, a multiple of 3 and the general term of progression of differences

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Abstract

This paper presents a new proof of Collatz conjecture using the classification in binary, a multiple of 3 and the general term of progression of differences. When Collatz process is done, we focus on numbers. Many sequences of numbers are generated. It is the progression of differences. The general term of progression of differences are computed. Then, the proof of contradiction, a multiple of 3 and the general term of progression of differences are used to prove that all positive odd other than one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process. Using the classification in binary, we focus on the number of digits. We calculate the expected values of digit (multiply 3 and add 1) (A) and (divide by 2) (B). Comparing the expected values of A and B, we find that there are unequal (B is greater than or equal to A). Thus, Collatz process does not diverge to positive infinity and eventually reaches one digit in binary. Since one digit obtained from Collatz process in binary is equal to 1 in decimal, number of times that the Collatz process reaches 1 is limited. Therefore, we clarify that Collatz conjecture is positive.

keywords: Collatz Conjecture, 3x+1, progresion of differences, expected value, digit, binary, classification, a multiple of 3

1 Introduction

A problem by Lothar Collatz in 1937, also called $3x+1$ mapping, $3x+1$ problem, kakutani's problem, syrucuse algorithm and Ulam's problem (Lagrarias 1985) [1]. The problem is simple. Pick a number, any positive integer. If it is odd, multiply it by 3 and add 1. If it is even, divide it by 2. Now you have a new number. Apply the same rules to the new number. The conjecture is about what happens as you keep repeating the process. Over the years, many mathematicians are trying to the proof of this conjecture. Terence Tao posted the proof of this conjecture for " almost all numbers " in 2022 [2]. But there is no proof that Collatz conjecture is positive or negative. Now we report the new proof using the binary and the general term of progression of differences, to clarify whether Collatz conjecture is positive or negative. If selected number is even then it reaches to odd by Collatz process. So, we take positive odd mainly to this proof.

2 First proof

The proof of contradiction and the general term of progression of differences are used to prove that all positive odd other than one do not enter an infinite loop (e.g.$1 \to 3 \to 4 \to 1$) by one cycle of Collatz process. For 2 or more cycles of Collatz process, proof is shown later.

2.1 Regularities in the numbers

Regularities in the numbers are derived from Collatz process. If we choose one, multiply one by three, add one and divide it by two (Collatz process), the choosed one is converged to one, the same for 5, 21, 85, 341, $\cdots$. If we choose three and calculate it using the Collatz process, we get the results that three is converged to five, the same for 13, 53, 213, 853, $\cdots$. It is the progression of differences. Table 1 and 2 show the results of Collatz process. The odd column is described odd from 1 to 15.

Table 1 Results of Collatz process from first to third term of sequence

Positive integer shown as k	Odd represented by 2k+1	Convergent type	First term of sequence	Second term of sequence	Third term of sequence
0	1	1	1	5	21
1	3	5	3	13	53
2	5	1	5	21	85
3	7	11	7	29	117
4	9	7	9	37	149
5	11	17	11	45	181
6	13	5	13	53	213
7	15	23	15	61	245
Moreover	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 2 Results of Collatz process from fourth to fifth term of sequence and moreover

Positive integer shown as k	Odd represented by 2k+1	Convergent type	Fourth term of sequence	Fifth term of sequence	Moreover
0	1	1	85	341	$\cdots$
1	3	5	213	853	$\cdots$
2	5	1	341	1365	$\cdots$
3	7	11	469	1877	$\cdots$
4	9	7	597	2389	$\cdots$
5	11	17	725	2901	$\cdots$
6	13	5	853	3413	$\cdots$
7	15	23	981	3925	$\cdots$
Moreover	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

2.2 The general term of progression of differences

We calculate the general term of progression of differences using the numbers from Collatz process. Odd numbers of 3, 13, 53, 213, 853, $\cdots$ from line second in table 1 and 2 are converged to five by Collatz process. This is the progression of differences, $a_n = a_1 + 10\sum_{k=1}^{n-1} 4^{k-1}$. The first term of sequence is three and common ratio is four. General term is shown below.

$$a_n = 3 + \frac{10(4^{n-1} - 1)}{3} \tag{1}$$

(Here n represents a natural number more than two and n holds even if one.)

Similarly odd numbers of 1, 5, 21, 85, 341, $\cdots$ and 7, 29, 117, 469, 1877, $\cdots$ from line first and fourth in table 1 and 2 are converged to one and eleven by Collatz process. General terms of sequence are shown below.

Odd numbers of 1, 5, 21, 85, 341, $\cdots$ from line first in table 1 and 2

$$a_n = 1 + \frac{4(4^{n-1} - 1)}{3} \tag{2}$$

(Here n represents a natural number more than two and n holds even if one.)

Odd numbers of 7, 29, 117, 469, 1877, $\cdots$ from line fourth in table 1 and 2

$$a_n = 7 + \frac{22(4^{n-1} - 1)}{3} \tag{3}$$

(Here n represents a natural number more than two and n holds even if one.)

About the all positive odd, we can represent as 2k+1 (k is a positive integer including zero). Then, the first term of sequence is 2k+1 and common retio is four. The general term of progression of differences is shown below.

$$a_n = (2k+1) + \frac{2(3k+2)(4^{n-1}-1)}{3} \tag{4}$$

(Here n represents a natural number more than two and n holds even if one.)

Prove that equation No.4 holds true for k (k is a positive integer including zero).

i) When k=0, equation No.4 is shown below.

$$a_n = 1 + \frac{4(4^{n-1}-1)}{3} \tag{5}$$

(Here n represents a natural number more than two and n holds even if one.)

ii) When k=1, equation No.4 is shown below.

$$a_n = 3 + \frac{10(4^{n-1}-1)}{3} \tag{6}$$

(Here n represents a natural number more than two and n holds even if one.)

iii) When equation No.6 - equation No.5, a result is shown below.

$$a_n = 2 \times 4^{n-1} \tag{7}$$

(Here n represents a natural number more than two and n holds even if one.)

iv) Assume k = g (g is a positive integer including zero) holds true. The equation No.4 becomes the following equation.

$$a_n = (2g+1) + \frac{2(3g+2)(4^{n-1}-1)}{3} \tag{8}$$

(Here n represents a natural number more than two and n holds even if one.)

v) When k = g + 1, equation No.4 becomes the following equation.

$$a_n = (2g+3) + \frac{2(3g+5)(4^{n-1}-1)}{3} \tag{9}$$

(Here n represents a natural number more than two and n holds even if one.)

vi) When equation No.9 - equation No.8, a result is shown below.

$$a_n = 2 \times 4^{n-1} \tag{10}$$

(Here n represents a natural number more than two and n holds even if one.)

From iii) and vi), $(No.6 - No.5) = (No.9 - No.8)$. When k=g and k=g+1, we know that the equation No.4 holds true. We are able to state that the equation No.4 holds true for all positive integer including zero by mathematical induction method. Since we describe the all positive odd as 2k+1 (k is a positive integer including zero), No.4 holds true for all positive odd.

Let multiply the No.4 by three and add one, the No.4 will become the No.11. No.11 is shown below.

$$a_n = 2(3k + 2) \times 4^{n-1} \tag{11}$$

(Here n represents a natural number more than two and n holds even if one. k is a positive integer including zero.)

No.11 is even and just before divide by 2.

2.3 Proposition

Prove that all positive odd other than one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by one cycle of Collatz process.

2.4 Proof

Assume, for the sake of contradiction, that two or more positive odd numbers including one enter infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by collatz process. Note that there exists the number returning to original odd same as one. It is unknown but we can be written as 2m+1, where m is the positive integer including zero.

As described the subsection 2.2, No.4 is proved to be true for all positive odd, then No.11 just before dividing by 2 is true for all positive odd. In accordance with this, No.11 has a number returning to original odd. In this sense, we can get the original odd dividing No.11 by 2 to enter an infinite loop. Then, this original odd is the same as 2m+1.

Let divide No.11 by 2 repeatedly until appearance of odd number. An odd obtained by dividing 2 is the equal to original odd number. If k is odd, then No.11 becomes 3k+2(this is odd). So, we show the equation 3k+2 = 2m+1(No.26). Even though there are many solutions in No.26, there is no solution that k and m are the same in positive integer including zero.

If k is even, then k is classified two types. For the first type, k is zero and $odd \times 2^n$, when n represents a natural number and $n \geq 2$. For the second type, k is $odd \times 2^n$, when n=one. In the first type, No.11 becomes $\frac{3}{2}k+1$(this is odd). So, we show that the equation $\frac{3}{2}k+1 = 2m+1$(No.27). Even though there are many solutions in No.27, only zero is the solution that k and m are the same in positive integer including zero. In the second type, No.11 becomes $\frac{3}{4}k+\frac{1}{2}$(this is odd). So, we show the equation $\frac{3}{4}k+\frac{1}{2} = 2m+1$(No.28). Even though there are many solutions in No.28, there is no solution that k and m are the same in positive integer including zero.

From the above, only zero is the solution that k and m are the same in positive integer including zero. So, there exists the only one infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process.

This is the contradiction.

Only one enters an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process, but all positive odd other than one do not enter an infinite loop. Therefore, it must be true that all positive odd except one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by one cycle of Collatz process. For 2 or more cycles of Collatz process, proof is shown next.

3 Second proof

A multiple of three and a classification of odd in binary are used to prove that all positive odd except one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by 2 or more cycles of Collatz process.

3.1 A multiple of three

The number after the Collatz process does not contain multiples of three. The Collatz process is an operation in which, when a number is chosen, if it is odd, it is multiplied by three and then add one to make it an even number, and then divided by two until it becomes an odd number. If an odd number is expressed as 2n+1 (n is a positive integer containing 0), then when multiplied by three, it becomes 6n+3 (n is a positive integer containing 0), and when one is added to it, it becomes 6n+4 (n is a positive integer containing 0). Since 6n+4=2(3n+2), we know that we can determine whether 3n+2 (n is a positive integer containing 0) is a multiple of three or not. Since 3n+2 (n is a positive integer including 0) is a multiple of three plus two when n is odd, it is not a multiple of three. When n is even, n is the form $odd \times 2^n$, and 1) n takes one or 2) n takes two or more. In the case of 2), 3n+2 (n is a positive integer containing 0) can be divided by two again, so it takes the form a multiple of three plus one, which is not a multiple of three. In the case of 1), dividing {3n} by two results in an odd number, it can be further divided by two again since (odd+1) is an even number. In this case, 3n+2 can be expressed as 3x2x(2m+1)+2 (m is a positive integer including 0), which is divided by two to form 3m+2, which is also a multiple of three plus two and not a multiple of three. Therefore, it is proved that the number after the Collatz process does not contain multiples of three.

3.2 Classification using the last three digits in binary

All positive odd numbers before Collatz process are classified into the following four types using the last three digits in binary.

1. Type A: The last three digits in binary are of the type (001). The numbers belonging to this type are 1, 9, 17, 25, 33, $\cdots$. This number sequence can be expressed as 8n+1 (n is a positive integer including 0).

2. Type B: The last three digits in binary are of the type (011). The numbers belonging to this type are 3, 11, 19, 27, 35, $\cdots$. This number sequence can be expressed as 8n+3 (n is a positive integer including 0).

Type C: The last three digits in binary are of the type (101). The numbers belonging to this type are 5, 13, 21, 29, 37, $\cdots$. This number sequence can be expressed as 8n+5 (n is a positive integer including 0).

Type D: The last three digits of this type in binary are of the type (111). The numbers belonging to this type are 7, 15, 23, 31, 39, $\cdots$. The number sequence can be expressed as 8n+7 (n is a positive integer including 0).

3.3 Results of Collatz process in type A,B and D

Type A.

If we do the Collatz process on type A, we get (8n+1)x3+1=24n+4=4(6n+1). Dividing by two, this number becomes 6n+1 (n is a positive integer containing 0). The result of the formula and {n} are shown in the table below.

Table 3 Results of Collatz process on type A

n	0	1	2	3	4	5	6	7	$\cdots$
6n+1	1	7	13	19	25	31	37	43	$\cdots$

Type B.

If we do Collatz process on type B, we get (8n+3)x3+1=24n+10=2(12n+5). Divided by 2, this number becomes 12n+5 (n is a positive integer containing 0). The result of the formula and {n} are shown in the table below.

Table 4 Results of Collatz process on type B

n	0	1	2	3	4	5	6	7	$\cdots$
12n+5	5	17	29	41	53	65	77	89	$\cdots$

Type D.

If we do Collatz process on type D, we get (8n+7)x3+1=24n+22=2(12n+11). Divided by 2, this number becomes 12n+11 (n is a positive integer containing 0). The result of the formula and {n} are shown in the table below.

Table 5 Results of Collatz process on type D

n	0	1	2	3	4	5	6	7	⋯
12n+11	11	23	35	47	59	71	83	95	⋯

Summarized type A, B and D.

We have proved that the number after Collatz process does not contain multiples of three. Let assume that the number after Collatz process contain multiples of three. This number becomes 6n+3 (n is a positive integer containing 0). The result of the formula and {n} is shown in the table below.

Table 6 Numbers after Collatz process contain multiples of three

n	0	1	2	3	4	5	6	7	⋯
6n+3	3	9	15	21	27	33	39	45	⋯

The above is summarized in the following table.

Table 7 Summarized numbers after Collatz process contain multiples of three

n	0	1	2	3	4	5	6	7	⋯
6n+1	1	7	13	19	25	31	37	43	⋯
12n+5	5	17	29	41	53	65	77	89	⋯
12n+11	11	23	35	47	59	71	83	95	⋯
6n+3	3	9	15	21	27	33	39	45	⋯

From this table 7, it is clear that once n is determined, odd numbers up to 12n+5 at $\frac{n}{2}$ can be arranged in order from 1 when n is even, and odd numbers up to 12n+11 at $\frac{n-1}{2}$ can be arranged in order from 1 when n is odd.

That is, when n is even, the numbers 6n+1, 6n+3 and 12 $\frac{n}{2} + 5$=6n+5 can be arranged in order, and when n is odd, 6n+1, 6n+3 and 12 $\frac{n-1}{2} + 11$=6n+5 can be arranged in order.

3.4 Results of Collatz process in type C

Let consider the case of type C. In binary, type C is a sequence of numbers whose last three digits are (101). The number after the Collatz process is 3n+2 (n is a positive integer including 0). Note that the number may be further divided by 2 depending on the value of n. This number type is characterized by the last letter of (01) in binary. For all positive odd numbers, no matter how many times the (01) is repeated at the end of number in binary, the result of the Collatz process is the same as the odd number before (01) is repeated at the end. For example, 11 (1011 in binary), 45 (101101 in binary), 181 (10110101 in binary), 725 (1011010101 in binary), or 2901 (101101010101 in binary) will all become the same 17 by Collatz processing.

3.5 Proposition

Prove that all positive odd other than one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by 2 or more cycles of Collatz process.

3.6 Proof

All positive odd numbers before Collatz process can be classified into the four types (A,B,C and D) in binary.

We have shown that if all positive odd numbers are classified into four types by the last three digits of the binary system, and if we choose types A, B, and D among them for the Collatz process, once n is determined, the odd numbers after the Collatz process, together with multiples of three, can be arranged in order from one. Then, after the Collatz process for an odd number, there will be only one number in that odd number, and that odd number will not have the same odd number. For the sake of clarity, we have combined multiples of three, but the result is the same even if a multiple of three exists only for the first time and not for second or more subsequent times. Therefore, for odd numbers of types A, B, and D, there is no loop due to the Collatz process (with the exception of one) by 2 or more cycles of Collatz process. Plus, we have shown that C-type odd numbers are odd numbers that take the form of adding a series of (01) at the end in binary to odd A, B, and D types, and that they do not loop infinitely because they only overlap with odd numbers after Collatz's processing to A, B, and D types.

It is proved that all positive odd other than one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by 2 or more cycles of Collatz process

4 Third proof

It is proved that Collatz process does not diverge to positive infinity and the number of times the Collatz process reaches $\{1\}$ is limited.

4.1 Multiply in binary

Consider to multiply in binary. 3 in decimal is described 11 in binary. Pick a number, any positive integer, selected number is denoted as (1aaa$\cdots$aaa) in binary, where $\{a\}$ is 0 or 1 regardless of the number of digits. A number multiplied by 11 (3 in decimal) in binary is shown below.

$$(1aaa\cdots aaa) \times 11 =$$

$$(1aaa\cdots aaa) \times 10 + (1aaa\cdots aaa) = (1aaa\cdots aaa0) + (1aaa\cdots aaa) \quad (12)$$

This is increased one digit and added original number in binary.

4.2 Divide in binary

Consider to divide in binary. 2 in decimal is described 10 in binary. Pick a number, any positive even in decimal, selected number is denoted as (1aaa··· aa0) in binary, where {a} is 0 or 1 regardless of the number of digits. A number divided by 10 (2 in decimal) in binary is shown below.

$$(1aaa\cdots aa0) \div 10 = (1aaa\cdots aa) \tag{13}$$

The one {0} elimination, this is decreased one digit in binary.

4.3 Positive odd classification by first 3 digits

Let classify positive odd in binary. Pick a number, any positive odd in decimal, selected number can be classified into four types by judgment of first 3 digits in binary regardless of the number of digits. There exist only four types in binary and every type is shown below.

$(100aaa\cdots aa1),\ where\ \{a\}\ is\ 0\ or\ 1,\ (No.29)$
$(101aaa\cdots aa1),\ where\ \{a\}\ is\ 0\ or\ 1,\ (No.30)$
$(110aaa\cdots aa1),\ where\ \{a\}\ is\ 0\ or\ 1,\ (No.31)$
$(111aaa\cdots aa1),\ where\ \{a\}\ is\ 0\ or\ 1,\ (No.32)$
Positive odd of 3 digits or less in binary will be described later.

4.4 Multiply from No.29 to 32 by 11(3 in decimal) in binary

Let multiply from No.29 to 32 by 11(3 in decimal) in binary. Calculation results are shown below.

$No.29 \times 11$
$(100aaa\cdots aa1) \times 11 =$

$$(100aaa\cdots aa10) + (100aaa\cdots aa1) = (110aaa\cdots a(a+1)1) \tag{14}$$

$No.30 \times 11$
$(101aaa\cdots aa1) \times 11 =$

$$(101aaa\cdots aa10) + (101aaa\cdots aa1) = (111(a+1)aaa\cdots a(a+1)1) \tag{15}$$

$No.31 \times 11$
$(110aaa\cdots aa1) \times 11 =$

$$(110aaa\cdots aa10) + (110aaa\cdots aa1) = (1001aaaa\cdots a(a+1)1) \tag{16}$$

$No.32 \times 11$
$(111aaa\cdots aa1) \times 11 =$

$$(111aaa\cdots aa10) + (111aaa\cdots aa1) = (1010(a+1)aaa\cdots a(a+1)1) \tag{17}$$

Regardless of values of {a}, single or double increase in digits is seen from No.14 to No.17. No.29 is multiplied by 11(3 in decimal) and a product becomes No.14. Single increase in digits of No.14 is seen. No.30 is multiplied by 11(3 in decimal) and a product becomes No.15. Single or double increase in digits of No.15 is seen. Single or double increase is related that {a} is zero or one. No.31 and No.32 are multiplied by 11(3 in decimal) and products become No.16 and No.17. Double increase in digits of No.16 and No.17 are seen.

4.5 Calculate the expected value of increased digits

Let calculate the expected value of increased digits, when No.29, 30, 31 and 32 are multiplied by 11(3 in decimal) in binary. Appearance probability P for No.29, 30, 31 and 32 are equal to 0.25. As described the subsection 3.3, there are only four positive odd types in binary. In accordance with this, the each expected value of increased digits is shown below.

No.14 is $0.25 \times 1 = 0.25$

No.15 is $0.25 \times 1 = 0.25$ *or* $0.25 \times 2 = 0.5$ (*at a maximum*)

No.16 is $0.25 \times 2 = 0.5$

No.17 is $0.25 \times 2 = 0.5$

In all positive odd, the expected value of increased digits in binary is 1.75(0.25+0.5+0.5+0.5) at a maximum.

4.6 Consider to add one in binary

Consider to add one in binary. Pick a number, any positive odd integer, selected number is denoted as $(1aaa \cdots aaa1)$ in binary. If all of {a} is 1, then single increase in digits by adding one is seen. This appearance probability is related to digits number. When the number of digits is 4, probability of appearance is 1/8. Five of digits is same as 1/16. That probability can be neglected, where there are many digits for all positive odd in binary.

4.7 Positive odd classification by last 3 digits

Let classify positive odd in binary again. Pick a number, any positive odd in decimal, selected number can be classified into four types by judgment of last 3 digits in binary. There exist only four types in binary and every type is shown below.

(1aa···aa001), where {a} is 0 or 1, (No.33)

(1aa···aa011), where {a} is 0 or 1, (No.34)

(1aa···aa101), where {a} is 0 or 1, (No.35)

(1aa···aa111), where {a} is 0 or 1, (No.36)

Positive odd of three digits or less (8 or less in decimal) in binary are proved that each of these numbers do not diverge to positive infinity by collatz process and reach one.

4.8 Multiply by 11(3 in decimal) and add 1 in binary

Let multiply No.33, No.34, No.35 and No.36 by 11(3 in decimal) and add 1 in binary. Calculation results are below.

$$(No.33 \times 11) + 1 = (1aa \cdots aa001) \times 11 + 1 =$$

$$(1aa \cdots aa0010) + (1aa \cdots aa001) + 1 =$$

$$(1aa \cdots aaa011) + 1 = (1aa \cdots aaa100) \tag{18}$$

$$(No.34 \times 11) + 1 = (1aa \cdots aa011) \times 11 + 1 =$$

$$(1aa \cdots aa0110) + (1aa \cdots aa011) + 1 =$$

$$(1aa \cdots aa(a+1)001) + 1 = (1aa \cdots aa(a+1)010) \tag{19}$$

$$(No.35 \times 11) + 1 = (1aa \cdots aa101) \times 11 + 1 =$$

$$(1aa \cdots aa1010) + (1aa \cdots aa101) + 1 =$$

$$(1aa \cdots aa(a+1)111) + 1 = (1aa \cdots aa(a+1+1)000) \tag{20}$$

$$(No.36 \times 11) + 1 = (1aa \cdots aa111) \times 11 + 1 =$$

$$(1aa \cdots aa1110) + (1aa \cdots aa111) + 1 =$$

$$(1aa \cdots aa(a+1+1)101) + 1 = (1aa \cdots aa(a+1+1)110) \tag{21}$$

4.9 Divide by 10 (2 in decimal) repeatedly in binary

Let divide No.18, No19, No.20 and No.21 by 10 (2 in decimal) repeatedly in binary until appearance of odd number. Regardless of values of {a}, the decrease of digits is seen from No.18 to No.21 dividing by 10 (2 in decimal) in binary. Calculation results are shown below.

$$No.18 \div 10(2 \ in \ decimal) =$$

$$(1aa \cdots aaa100) \div 10 = \cdots = (1aaa \cdots aa1) \tag{22}$$

The two {0} elimination, this is decreased two digits.

$$No.19 \div 10(2\ in\ decimal) =$$

$$(1aa \cdots aa(a+1)010) \div 10 = (1aaa \cdots aa(a+1)01) \tag{23}$$

The one {0} elimination, this is decreased one digit.

$$No.20 \div 10(2\ in\ decimal) =$$

$$(1aa \cdots aa(a+1+1)000) \div 10 = \cdots = (1aaa \cdots aa(a+1+1)) \tag{24}$$

The three {0} elimination, this is decreased three digits. Depending on the values of {a} (0 or 1), three or more {0} are eliminated. Three or more of digits (3 of digits are at a minimum) are decreased.

$$No.21 \div 10(2\ in\ decimal) =$$

$$(1aa \cdots aa(a+1+1)110) \div 10 = (1aaa \cdots aa(a+1+1)11) \tag{25}$$

The one {0} elimination, this is decreased one digit.

4.10 Calculate the expected value of decreased digits

Let calculate the expected values of decreased digits. Appearance probability P for No.18, No.19, No.20 and No.21 are equal to 0.25. As described the subsection 3.7, there are only four positive odd types in binary. In accordance with this, the each expected value of decreased digits is shown below.

No.22 is $0.25 \times 2 = 0.5$

No.23 is $0.25 \times 1 = 0.25$

No.24 is $0.25 \times 3 = 0.75$ (*at a minimum*)

No.25 is $0.25 \times 1 = 0.25$

In all positive odd, the expected value of decreased digits is 1.75(0.5+0.25+0.75+0.25) at a minimum.

4.11 Proposition

All positive odd do not diverge to positive infinity by Collatz process and the number of times that the Collatz process reaches {1} is limited.

Pick a number, any positive integer. If it is odd, multiply it by 3 and add 1. If it is even, divide it by 2. Now you have a new number. Apply the same rules to the new number. This is the collatz process.

4.12 Proof

About all positive odd numbers, let compare the expected value of decreased digits by dividing 10 (2 in decimal) in binary (shown as B) with the expected value of increased digits by multiplying by 11 (3 in decimal) and adding one in binary (shown as A). Here, the expected value of increased digits by adding one can be neglected as described the subsection 3.6. A is equal to B as described the subsection 3.5 and 3.10. But B is at a minimum and A is at a maximum. In according with this, there is inequality (B is greater than or equal to A). Therefore, all positive odd do not diverge to positive infinity by Collatz process.

Although the number of digits may increase by Collatz process, the expected value of decreased digits is greater than or equal to the expected value of increased digits. The expected value is based on the law of large number. Then digits are decreased gradually by Collatz process in binary. Moreover, it is proved at the subsection 2.4 that all positive odd except one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$) by Collatz process. Thus, Collatz process does not diverge to positive infinity and eventually reaches the one digit in binary. Since one digit in binary is equal to $\{1\}$ in decimal, number of times that the Collatz process reaches $\{1\}$ is limited.

5 Discussion and Conclusion

As expressed in section 2 (First proof), 3 (Second proof) and 4 (Third proof), we have proved that all positive odd does not diverge to positive infinity by Collatz process and reach $\{1\}$ within the limited repeat times. And also, we have proved that all positive odd other than one do not enter an infinite loop (e.g. $1 \to 3 \to 4 \to 1$). Therefore, it is concluded that Collatz conjecture is positive. But we cannot calculate the number of repeated times of Collatz process in all positive odd. We consider that this is the remaining problem.

Proof of contradiction using the general term of sequence and calculation of the expected value by number classification in binary are very useful in this proof. The focus on digit in binary is a turning point of the proof about Collatz conjecture. Numbers in decimal and digits in binary are related but different concepts. It can be said that each is the different universe. We may be play the positive proof of Collatz conjecture in the "Hodge theater" [3].

6 Declaration

Author has no conflict of interest to declare.

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8 Refference

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