L-FUNCTIONS WITH THE LINDELÖF AND RIEMANN HYPOTHESES AND TRIVIAL ZEROS AT EVEN NEGATIVE INTEGERS

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ABSTRACT. Let $q \ge 2$ be an integer, $\zeta(s)$ be the Riemann zeta function, and put $T_q(s) := (s+1)(1-s)^{-1}(q^{s+2}-1)\zeta(s+2) - 4\pi^2 s^{-1}(1-s)^{-1}(q^{3-s}-1)\zeta(s-2)$. In the present paper, we show that the function $T_q(s)$ has Riemann's functional equation and its zeros only at the negative even integers and satisfies the Lindelöf and Riemann hypotheses. In addition, we give functions satisfy Riemann's functional equation and an analogue of the Lindelöf hypothesis but do not fulfill an analogue of the Riemann hypothesis.

1. Introduction and Main results

1.1. **Lindelöf and Riemann hypotheses.** Let $q > 2$ be an integer, and $\chi(n)$ be a Dirichlet character (mod q). Then, for $\Re(s) := \sigma > 1$, the Riemann zeta function $\zeta(s)$ and the Dirichlet L-function $L(s,\chi)$ are defined by the ordinary Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

respectively. The Riemann zeta function $\zeta(s)$ is continued meromorphically and has a simple pole at $s=1$ with residue 1. The Dirichlet L-function $L(s,\chi)$ can be analytically continued to the whole complex plane to a holomorphic function if $B_0(\chi) := \sum_{r=0}^{q-1} \chi(r)/q = 0$, otherwise to a meromorphic function with a simple pole, at $s=1$, with residue $B_0(\chi)$.

The distribution of zeros of the Riemann zeta function is one of the central problems in mathematics. By the Euler product of $\zeta(s)$, the Riemann zeta function does not vanish when $\sigma > 1$. In addition, $\zeta(s) \ne 0$ for $\Re(s) < 0$ except for $s = -2n$, where $n \in \mathbb{N}$ by the fact above and the functional equation $\zeta(s)$. The Riemann hypothesis (RH, in short) is concerned with the locations of nontrivial (non-real) zeros, and states that:

RH *The real part of every nontrivial zero of $\zeta(s)$ is $1/2$.*

The following estimation of the order of the Riemann zeta function is widely known (e.g. [19, Chapter 5]):

$$|t|^{1/2-\sigma} \ll \zeta(\sigma+it) \ll |t|^{1/2-\sigma}, \qquad |t| \ge 1, \quad \sigma < 0. \tag{1.1}$$

For each $\sigma \in \mathbb{R}$, we define $\mu(\sigma)$ as the lower bound of the numbers ξ such that

$$\zeta(\sigma+it) = O\bigl(|t|^{\xi}\bigr), \qquad |t| \ge 1.$$

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The Lindelöf hypothesis (LH, in short) is that the graph of $\mu(\sigma)$ is written by

$$\textbf{LH} \qquad \mu(\sigma) = \begin{cases} 1/2 - \sigma & \sigma \le 1/2, \\ 0 & \sigma > 1/2. \end{cases} \tag{1.2}$$

The following fact is well-known

the Lindelöf hypothesis is implied by the Riemann hypothesis.

Let $N(T,\zeta)$ denote the numbers of zeros of $\zeta(s)$ in the region $0 \le \Re(s) \le 1$ and $0 < \Im(s) < T$. Then the following Riemann-von Mangoldt formula is well-known (e.g., [19, Theorem 9.4]). As $T \to \infty$,

$$N(T,\zeta) = \frac{T}{2\pi}\log T - \frac{1+\log 2\pi}{2\pi} T + O(\log T).$$

Similar asymptotic formula holds for Dirichlet L-functions (e.g., [6, Theorem 5.8]). As an analogue of the Riemann hypothesis, the generalized Riemann hypothesis (GRH, in short) asserts that, for every Dirichlet character χ and every complex number $s \notin \mathbb{R}_{<0}$ with $L(s,\chi)=0$, then the real part of $s \in \mathbb{C}$ is 1/2.

Clearly, the Riemann hypothesis implies that the number of zeros of $\zeta(s)$ on the line segment from $1/2$ to $1/2+iT$ coincides with $N(T,\zeta)$. Related to this fact, Bombieri and Hejhal gave functions whose almost all non-real zeros are located only on $\Re(s)=1/2$ but do not satisfy an analogue of the Riemann hypothesis. More precisely, they showed that

$$\sum_{h=1}^{j} b_h L(s,\chi_h), \qquad j>1, \quad b_h \in \mathbb{R}\setminus\{0\}, \tag{1.3}$$

where χ_h ranges over distinct primitive even characters to some fixed modulus q (similarly if χ_h ranges over odd characters), has 100 percent of zeros on the line $\sigma = 1/2$ under the GRH and assumptions on well-spacing of zeros for Dirichlet L-functions (see [2, Theorem A]). Note that the function $\sum_{h=1}^{j} b_h L(s,\chi_h)$ has infinitely many non-real zeros in the strip $1/2 < \Re(s) < 1$ by [7, Theorem 7.3] or [17, Theorem].

Neither the RH nor GRH are proved. However, there are some functions whose all non-real zeros are located only on the critical line $\Re(s)=1/2$. For example, Taylor [18, Section 1] showed that

$$\zeta^*(s+1/2) - \zeta^*(s-1/2), \qquad \zeta^*(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

has all its zeros on the critical line $\sigma=1/2$ (see also [12, Exercise 10.2.1.7]). His theorem, which can be proved by Proposition 2.2, is generalized by many mathematicians, for example, Lagarias & Suzuki [10] and Müller [13]. On the contrary, there are no examples which satisfy an analogue of (1.1) and the (1.2) until now (see Section 3.1).

1.2. **Trivial zeros and Riemann's functional equation.** It is widely known that $\zeta(s)$ satisfies Riemann's functional equation

$$\zeta(1-s) = \Gamma_{\cos}(s)\zeta(s), \qquad \Gamma_{\cos}(s) := \frac{2\Gamma(s)}{(2\pi)^s}\cos\Bigl(\frac{\pi s}{2}\Bigr) \tag{1.4}$$

(e.g., [19, (2.1.8)]). The first converse theorem on the Riemann zeta function $\zeta(s)$ is proved by Hamburger [4, Satz 1] (see also [19, Chapter 2.13]) who characterized $\zeta(s)$ by Riemann's functional equation above.

Hamburger's theorem has many analogies and generalizations but there are also studies that give "counterexamples" by weakening some assumptions in his theorem. The first counterexamples are given by Koshliakov [9]. Let $r > 0$ and $\lambda_1, \lambda_2, \lambda_3, \ldots$ be the positive roots of the equation $r \sin(\pi\lambda) + \lambda \cos(\pi\lambda) = 0$. Then, for $\Re(s) > 1$, Koshliakov [9] defined the function

$$\theta_r(s) := \sum_{j=1}^{\infty} \frac{r^2 + \lambda_j^2}{r(r+\pi^{-1}) + \lambda_j} \frac{1}{\lambda_j^s}.$$

Furthermore, Koshliakov defined the function

$$\eta_r(s) := \sum_{k=1}^{\infty} \frac{(s, 2\pi r k)_k}{k^s}, \qquad (s, \nu k)_k := \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-x} \left(\frac{k\nu - x}{k\nu + x} \right)^k x^{s-1} dx.$$

Koshliakov showed that $\theta_r(1-s) = \Gamma_{\cos}(s)\eta_r(s)$ and $\eta_r(1-s) = \Gamma_{\cos}(s)\theta_r(s)$ which imply Riemann's functional equation

$$\omega_r(1-s) = \Gamma_{\cos}(s)\omega_r(s), \qquad 2\omega_r(s) := \eta_r(s) + \theta_r(s)$$

(see also [3, Sections 2 and 4]). Without mentioning Koshliakov's paper, Knopp [8] proved that there are infinitely many linearly independent solutions if we relax Hamburger's or Hecke's condition on poles in their converse theorems. It should be emphasised that Knopp gives no explicit representation for the solutions satisfying Riemann's functional equation. In addition, there are no paper on non-trivial zeros (namely, RH) and the estimation of the order (namely, LH) of Koshliakov's and Knopp's functions.

Owing to Riemann's functional equation, the Riemann zeta function $\zeta(s)$ has trivial zeros at even negative integer. Note that some functions satisfy functional equations but do not fulfill Riemann's functional equation, namely, these functions do not vanish at even negative integers. For example, the Davenport-Heilbronn function satisfy a functional equation but does not fulfill Riemann's functional equation since the function has real zeros at negative odd integers (see [19, Chapter 10.25]). Recall that Taylor's function satisfies an analogue of the Riemann hypothesis (see Section 1.1). Changing the variable $s = 2\tilde{s} - 1/2$, which maps the critical line to itself, this asserts $F(\tilde{s}) := \zeta^*(2\tilde{s}) - \zeta^*(2\tilde{s}-1)$, satisfies the RH and the functional equation

$$F(\tilde{s}) = -F(1-\tilde{s}),$$

which is proved by $\zeta^*(2\tilde{s}-1) = \zeta^*(2-2\tilde{s})$. It should be emphasized that the function $F(\tilde{s})$ does NOT satisfy Riemann's functional equation

$$F(1-\tilde{s}) = \Gamma_{\cos}(\tilde{s})F(\tilde{s}).$$

Therefore, the function $F(\tilde{s})$ satisfies an analogue of the RH but does not have trivial zeros at negative even integers (see [10, Section 1]).

However, the author [16, Theorem 1.1] constructed functions with Riemann's functional equation and the Riemann hypothesis. Let χ_3 and χ_4 be the non-principal Dirichlet characters mod 3 and 4, respectively and define $R_1(s)$ and $R_2(s)$ by

$$R_1(s) := s3^{s+1}L(s+1, \chi_3) + 2\pi\sqrt{3}L(s-1, \chi_3),$$
$$R_2(s) := s4^{s+1}L(s+1, \chi_4) + 4\pi L(s-1, \chi_4).$$

Then, the functions $R_1(s)$ and $R_2(s)$ fulfill Riemann's functional equation

$$R_1(1-s) = \Gamma_{\cos}(s)R_1(s), \qquad R_2(1-s) = \Gamma_{\cos}(s)R_2(s).$$

In addition, all non-real zeros of $R_1(s)$ and $R_2(s)$ are on the critical line $\sigma = 1/2$, and analogues of Riemann-von Mangoldt formulas hold for $R_1(s)$ and $R_2(s)$. We remark that the gamma factor in Riemann's functional equations above completely coincide with that of these functional equation for $\zeta(s)$ appearing in (1.4) and the function $R_1(s)$ and $R_2(s)$ have trivial zeros at non-positive even integers. Moreover, it should be emphasised that Riemann's functional equation for $R_1(s)$ and $R_2(s)$ does not contradict to Hamburger's theorem since the functions can not be expressed as any Dirichlet series.

1.3. **Main results.** In the present paper, we generalize the functions $R_1(s)$ and $R_2(s)$ above. Let χ be a primitive real Dirichlet character mod q and put

$$R^{(l)}(s,\chi) := (s)_l q^{s+l} L(s+l,\chi) + (2\pi)^l \psi(l) \sqrt{q} L(s-l,\chi),$$

where $l \in \mathbb{N}$, $(s)_l$ and $\psi(l)$ are defined as

$$(s)_l := s(s+1)\cdots(s+l-1), \qquad \psi(l) := \begin{cases} 1 & l \equiv 0,1 \mod 4, \\ -1 & l \equiv 2,3 \mod 4. \end{cases}$$

Then we have the following.

Theorem 1.1. *Let l be an odd natural number and χ be odd. Then, $R^{(l)}(s,\chi)$ satisfies Riemann's functional equation*

$$R^{(l)}(1-s,\chi) = \Gamma_{\cos}(s) R^{(l)}(s,\chi) \tag{1.5}$$

with $R^{(l)}(1/2,\chi) > 0$, has its zeros only at the non-positive even integers and non-real numbers with real part $1/2$. Moreover, when l is an even natural number and χ is an even Dirichlet character, the function $R^{(l)}(s,\chi)$ has the same property.

For principal Dirichlet characters, we have the following.

Theorem 1.2. *Let q be a natural number greater than 1 and put*

$$\zeta_q^{(2k)}(s) := (s)_{2k}\bigl(q^{s+2k}-1\bigr)\zeta(s+2k) + (-1)^k (2\pi)^{2k}\bigl(q^{1-s+2k}-1\bigr)\zeta(s-2k).$$

Then, the function $\zeta_q^{(2k)}(s)$ has Riemann's functional equation $\zeta_q^{(2k)}(1-s) = \Gamma_{\cos}(s)\zeta_q^{(2k)}(s)$ with $\zeta_q^{(2k)}(1/2) > 0$, and its zeros only at the non-positive even integers and non-real numbers with real part $1/2$.

By theorems above, we have the following main result. It should be emphasised that the pole and real zeros of the functions $U(s,\chi)$ and $T_q(s)$ below are located only on $s = 1$ and $s/2 \in \mathbb{Z}_{\le -1}$ just like the Riemann zeta function. Furthermore, the functions $U(s,\chi)$ and $T_q(s)$ satisfy an analogue of (1.1) and (1.2) (see Section 3.1).

Theorem 1.3. *Let χ be a primitive real even Dirichlet character mod q, where q is a natural number greater than 1. We put*

$$U(s,\chi) := \frac{R^{(2)}(s,\chi)}{s(1-s)}, \qquad T_q(s) := \frac{\zeta_q^{(2)}(s)}{s(1-s)}.$$

Then, the function $U(s,\chi)$ has Riemann's functional equation $U(1-s,\chi) = \Gamma_{\cos}(s)U(s,\chi)$ with $U(1/2,\chi) > 0$, a pole at $s = 1$ and its zeros only at the negative even integers and non-real numbers with real part $1/2$. Furthermore, one has

$$|t|^{1/2-\sigma} \ll U(\sigma+it,\chi) \ll |t|^{1/2-\sigma}, \qquad |t| \ge 1, \quad \sigma < 0. \tag{1.6}$$

Moreover, the number $\mu(\sigma)$ defined as the lower bound of the numbers ξ such that $U(\sigma+it,\chi) \ll_{\sigma,q} |t|^\xi$ is given by (1.2). In addition, the function $T_q(s)$ has the same property.

As an analogue of (1.3), we consider the following function. Let $b_h \in \mathbb{C}\setminus\{0\}$ and χ_h be a real Dirichlet character mod q and put

$$R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b}) := \sum_{h=1}^{j} b_h R^{(l)}(s,\chi_h). \tag{1.7}$$

Then we have the following. Note that when $j=1$, we can take $l_0=1$ in the statement below by Theorem 1.1.

Proposition 1.4. *Suppose $b_1,\dots,b_j>0$ and all Dirichlet characters $\chi_1,\dots,\chi_j$ are odd. Then the function $R^{(2l-1)}(s,\boldsymbol{\chi},\boldsymbol{b})$ satisfies Riemann's functional equation*

$$R^{(2l-1)}(1-s,\boldsymbol{\chi},\boldsymbol{b}) = \Gamma_{\cos}(s) R^{(2l-1)}(s,\boldsymbol{\chi},\boldsymbol{b}). \tag{1.8}$$

Furthermore, there exist $l_0\in\mathbb{N}$ such that for any $l\ge l_0$, the function $R^{(2l-1)}(s,\boldsymbol{\chi},\boldsymbol{b})$ has its zeros only at the non-positive even integers and complex numbers with real part $1/2$.

When all $\chi_1,\dots,\chi_j$ are even, the same statement holds if $2l-1$ is replaced by $2l$.

The theorem above should be compared with the fact that $\sum_{h=1}^{j} b_h L(s,\chi_h)$ does not satisfy analogues of the GRH (see Section 1.1). In addition, we have the following which gives functions satisfy an analogue of the Lindelöf hypothesis but do not fulfill an analogue of the Riemann hypothesis.

Theorem 1.5. *Let $\chi_1,\dots,\chi_j$ be primitive real even distinct Dirichlet characters, assume that there exists $m\in\mathbb{N}$ such that $\sum_{h=1}^{j} b_h\chi_h(m)\neq 0$, where $b_h\in\mathbb{C}\setminus\{0\}$, and put*

$$U(s,\boldsymbol{\chi},\boldsymbol{b}) := \frac{R^{(2)}(s,\boldsymbol{\chi},\boldsymbol{b})}{s(1-s)}.$$

Then, the function $U(s,\boldsymbol{\chi},\boldsymbol{b})$ has a pole at $s=1$ and Riemann's functional equation $U(1-s,\boldsymbol{\chi},\boldsymbol{b}) = \Gamma_{\cos}(s)U(s,\boldsymbol{\chi},\boldsymbol{b})$. Furthermore, the number $\mu(\sigma)$ defined as the lower bound of the numbers ξ such that $U(\sigma+it,\boldsymbol{\chi},\boldsymbol{b}) \ll_{\sigma,q} |t|^\xi$ is given by (1.2). However, there exist $j\in\mathbb{N}$ and $b_1,\dots,b_j\in\mathbb{C}\setminus\{0\}$ such that $U(s,\boldsymbol{\chi},\boldsymbol{b})$ has a zero for $\Re(s)>1/2$.

In addition, we have the Riemann von Mangoldt formula for $R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b})$.

Proposition 1.6. *Let χ be a real Dirichlet character mod q. Then we have*

$$N\big(T, R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b})\big) = \frac{T}{2\pi}\log T + \frac{\log(q^2/2\pi)-1}{2\pi}T + O(\log T). \tag{1.9}$$

The contents of the paper are as follows. In Section 2, we give proofs of results in this subsection. In Section 3, we give some remarks on the Lindelöf hypothesis, infinite product representations, Hardy's Z-functions and numerical calculations for $T_2(s)$ and $U(s,\chi_5)$, where χ_5 is the non-primitive real Dirichlet character mod 5.

2. Proof

2.1. **Preliminaries.** For $0<a\le 1$, we define the Hurwitz zeta function $\zeta(s,a)$ and the periodic zeta function $F(s,a)$ by

$$\zeta(s,a) := \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \qquad F(s,a) := \sum_{n=1}^{\infty}\frac{e^{2\pi i n a}}{n^s}, \qquad \sigma>1,$$

respectively. The both infinite series of $\zeta(s,a)$ and $F(s,a)$ converge absolutely in the region $\sigma > 1$ and uniformly in each compact subset of this half-plane. The Hurwitz zeta function $\zeta(s,a)$ can be analytically continued for all $s \in \mathbb{C}$ except $s = 1$, where there is a simple pole with residue 1 (e.g., [1, Section 12]). If $0 < a < 1$, the Dirichlet series of $F(s,a)$ converges uniformly in each compact subset of the half-plane $\sigma > 0$ (e.g., [11, p. 20]) and $F(s,a)$ is continuable to the whole complex plane (e.g., [11, Chapter 2.2]).

Next, for $0 < a \le 1/2$, define functions $Z(s,a)$, $P(s,a)$, $Y(s,a)$ and $O(s,a)$ by

$$Z(s,a) := \zeta(s,a) + \zeta(s,1-a), \qquad P(s,a) := F(s,a) + F(s,1-a)$$
$$Y(s,a) := \zeta(s,a) - \zeta(s,1-a), \qquad O(s,a) := -i\bigl(F(s,a) - F(s,1-a)\bigr).$$

We can easily see that $P(s,a)$, $Y(s,a)$ and $O(s,a)$ are entire functions when $0 < a < 1/2$. However, the function $Z(s,a)$ has a simple pole at $s = 1$. In addition, for $0 < a \le 1/2$, define the quadrilateral zeta function $Q(s,a)$ by

$$2Q(s,a) := \zeta(s,a) + \zeta(s,1-a) + F(s,a) + F(s,1-a).$$

Clearly, we have $2Q(s,a) = Z(s,a) + P(s,a)$. The function $Q(s,a)$ can be continued analytically to the whole complex plane except $s = 1$. In [15, Theorem 1.1], the author prove the functional equation

$$Q(1-s,a) = \Gamma_{\cos}(s) Q(s,a). \tag{2.1}$$

We remark that the gamma factor in (2.1) completely coincides with that of the functional equation for $\zeta(s)$ appearing in (1.4). Note that Riemann's functional equation for $Q(s,a)$ does not contradict to Hamburger's theorem since $Q(s,a)$ can not be expressed as any ordinary Dirichlet series.

In [14, Proposition 2.2], the following is proved. Let χ be a primitive Dirichlet character, $G(\chi)$ be the Gauss sum associated to χ, and $0 < r < q$ be relatively prime integers. If $\chi(-1) = 1$, we have

$$L(s,\chi) = \frac{1}{2q^s} \sum_{r=1}^{q} \chi(r) Z(s, r/q) = \frac{1}{2G(\overline{\chi})} \sum_{r=1}^{q} \overline{\chi}(r) P(s, r/q). \tag{2.2}$$

When $\chi(-1) = -1$, one has

$$L(s,\chi) = \frac{1}{2q^s} \sum_{r=1}^{q} \chi(r) Y(s, r/q) = \frac{i}{2G(\overline{\chi})} \sum_{r=1}^{q} \overline{\chi}(r) O(s, r/q). \tag{2.3}$$

Let χ be a real primitive character modulo q, where $q > 1$ and $2\kappa(\chi) := 1 - \chi(-1)$. Then, it is widely known (e.g., [6, Chapter 4.6]) that

$$\xi(s,\chi) = \xi(1-s,\chi), \qquad \xi(s,\chi) = \left(\frac{q}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{s+\kappa(\chi)}{2}\right) L(s,\chi). \tag{2.4}$$

From [6, Theorem 5.6], we can see that $\xi(s,\chi)$ is an entire function of genus one when χ is primitive. According to Phragmèn-Lindelöf convexity principal and the functional equation above, we have the following (see [19, Chapter 5.1]).

Lemma 2.1. *Let $\varepsilon > 0$. Then we have*

$$\zeta(s),\ L(s,\chi) \ll_{q,\sigma} |t|^{g_\varepsilon(\sigma)}, \qquad g_\varepsilon(\sigma) := \begin{cases} 1/2 - \sigma & \sigma < 0, \\ (1-\sigma)/2 + \varepsilon & 0 \le \sigma \le 1, \\ 0 & \sigma > 1. \end{cases}$$

Taylor's theorem mentioned in Section 1.1 can be proved by the following shown by Lagarias and Suzuki [10]. It should be noted that their theorem is a key for the proof of the Riemann hypothesis for $R^{(l)}(s,\chi)$ and $\zeta_q^{(2k)}(s)$.

Lemma 2.2 ([10, Theorem 4])**.** *Let $F(s)$ be an entire function of genus zero or one, be real on the real axis, and satisfy $F(s) = \pm F(1-s)$ for some choice of sign, and there exists $\alpha > 0$ such that all zeros of $F(s)$ lie in the vertical strip $|\Re(s) - 1/2| < \alpha$.*

Then for any real $\gamma \ge \alpha$,

$$\left| \frac{F(s+\gamma)}{F(s-\gamma)} \right| > 1 \quad \textit{if} \quad \Re(s) > \frac{1}{2}, \qquad \left| \frac{F(s+\gamma)}{F(s-\gamma)} \right| < 1 \quad \textit{if} \quad \Re(s) < \frac{1}{2}.$$

The next proposition is easily proved by the lemma in [19, Section 9.4].

Lemma 2.3. *Let $\sigma_1 > 1$ and $0 \le \alpha \le \beta < \sigma_1$. Let $f(s)$ be an analytic function, real for real s, regular for $\sigma \ge \alpha$; let $|\Re(f(\sigma_1 + it))| \ge m > 0$ and*

$$\bigl| f(\sigma' + it') \bigr| < M_{\sigma,t}, \qquad \sigma' \ge \sigma, \quad 1 \le t' \le t.$$

Then if T is not the ordinate of a zero of $f(s)$,

$$\bigl| \arg f(\sigma + iT) \bigr| < \frac{\pi}{\log((\sigma_1 - \alpha)/(\sigma_1 - \beta))} \bigl(\log M_{\alpha, T+2} - \log m \bigr) + \frac{3\pi}{2}, \qquad \sigma \ge \beta.$$

2.2. **Proof of Theorem 1.1.** Roughly speaking, we prove the functional equation and Riemann hypothesis for $R^{(l)}(s,\chi)$ by the equality (2.1) and Lemma 2.2, respectively.

Proof of Riemann's functional equation for $R^{(l)}(s,\chi)$. For $l \in \mathbb{N}$, we have

$$S^{(l)}(1-s,a) = \Gamma_{\cos}(s) S^{(l)}(s,a), \qquad S^{(l)}(s,a) := 2 \frac{\partial^l}{\partial^l a} Q(s,a) \tag{2.5}$$

by differentiating both side of (2.1) with respect to $0 < a \le 1/2$. For $\sigma > 3$, it holds that

$$\frac{\partial}{\partial a} Z(s,a) = \sum_{n=0}^{\infty} \frac{\partial}{\partial a} \left(\frac{1}{(n+a)^s} + \frac{1}{(n+1-a)^s} \right) = -sY(s+1,a),$$

$$\frac{\partial}{\partial a} P(s,a) = \sum_{n=0}^{\infty} \frac{\partial}{\partial a} \frac{\cos(2\pi n a)}{n^s} = -2\pi \sum_{n=0}^{\infty} \frac{\sin(2\pi n a)}{n^{s-1}} = -2\pi O(s-1,a).$$

Moreover, one has

$$\frac{\partial^2}{\partial^2 a} Z(s,a) = -s \frac{\partial}{\partial a} Y(s+1,a) = s(s+1) Z(s+2,a),$$

$$\frac{\partial^2}{\partial^2 a} P(s,a) = -2\pi \frac{\partial}{\partial a} O(s-1,a) = -(2\pi)^2 P(s-2,a).$$

Thus, for $l = 2k-1$, we have

$$\begin{aligned} Z^{(2k-1)}(s,a) &:= \frac{\partial^{2k-1}}{\partial^{2k-1} a} Z(s,a) = (-1)^{2k-1} (s)_{2k-1} Y(s+2k-1,a), \\ P^{(2k-1)}(s,a) &:= \frac{\partial^{2k-1}}{\partial^{2k-1} a} P(s,a) = (-1)^k (2\pi)^{2k-1} O(s-2k+1,a), \end{aligned} \tag{2.6}$$

if $\Re(s) > 0$ is sufficiently large. When $l = 2k$, it holds that

$$\begin{aligned} Z^{(2k)}(s,a) &:= \frac{\partial^{2k}}{\partial^{2k} a} Z(s,a) = (-1)^{2k} (s)_{2k} Z(s+2k,a), \\ P^{(2k)}(s,a) &:= \frac{\partial^{2k}}{\partial^{2k} a} P(s,a) = (-1)^{k} (2\pi)^{2k} P(s-2k,a), \end{aligned} \tag{2.7}$$

for sufficiently large $\Re(s) > 0$. Recall that $Z(s,a)$, $Y(s,a)$, $P(s,a)$ and $O(s,a)$ are continued meromorphically. By this fact, equations in (2.6) and (2.7) holds for all $s \in \mathbb{C}$ except for singularities.

We have $iG(\chi) = \sqrt{q}$ by [6, Theorem 3.3] if χ is odd and real. Hence, we obtain

$$\begin{aligned} &\sum_{r=1}^{q-1} \chi(r) S^{(2k-1)}(s,r/q) = \sum_{r=1}^{q-1} \chi(r) Z^{(2k-1)}(s,r/q) + \sum_{r=1}^{q-1} \chi(r) P^{(2k-1)}(s,r/q) \\ &= (-1)^{2k-1} (s)_{2k-1} \sum_{r=1}^{q-1} \chi(r) Y(s+2k-1,r/q) + (-1)^k (2\pi)^{2k-1} \sum_{r=1}^{q-1} \chi(r) O(s-2k+1,r/q) \\ &= (-1)^{2k-1} (s)_{2k-1} q^{s+2k-1} L(s+2k-1,\chi) + (-1)^k (2\pi)^{2k-1} \sqrt{q} L(s-2k-1,\chi) \end{aligned}$$

from (2.3) and (2.6). On the other hand, one has $G(\chi) = \sqrt{q}$ by [6, Theorem 3.3] when χ is even and real. Thus, by (2.2) and (2.7), we have

$$\begin{aligned} &\sum_{r=1}^{q-1} \chi(r) S^{(2k)}(s,r/q) = \sum_{r=1}^{q-1} \chi(r) Z^{(2k)}(s,r/q) + \sum_{r=1}^{q-1} \chi(r) P^{(2k)}(s,r/q) \\ &= (-1)^{2k} (s)_{2k} \sum_{r=1}^{q-1} \chi(r) Z(s+2k,r/q) + (-1)^k (2\pi)^{2k} \sum_{r=1}^{q-1} \chi(r) P(s-2k,r/q) \\ &= (s)_{2k} q^{s+2k} L(s+2k,\chi) + (-1)^k (2\pi)^{2k} \sqrt{q} L(s-2k,\chi). \end{aligned}$$

Clearly, we have

$$\sum_{r=1}^{q-1} \chi(r) S^{(l)}(1-s,r/q) = \Gamma_{\cos}(s) \sum_{r=1}^{q-1} \chi(r) S^{(l)}(s,r/q)$$

from (2.5). In addition, one has

$$(-1)^{2k-1} R^{(2k-1)}(s,\chi) = \sum_{r=1}^{q-1} \chi(r) S^{(2k-1)}(s,r/q), \qquad R^{(2k)}(s,\chi) = \sum_{r=1}^{q-1} \chi(r) S^{(2k)}(s,r/q).$$

Therefore, we obtain Riemann's functional equation for $R^{(l)}(s,\chi)$. □

Proof of the Riemann hypothesis for $R^{(2k-1)}(s,\chi)$. Assume $l = 2k-1$ and χ is odd. Define the function $\Gamma^{\flat}_{k,q}(s)$ as

$$\Gamma^{\flat}_{k,q}(s) := \left(\frac{q}{\pi}\right)^{s/2-k+1} \Gamma\left(\frac{s}{2} - k + 1\right).$$

By the well-known formula $s\Gamma(s) = \Gamma(s+1)$ and the definition of $\xi(s,\chi)$, one has

$$\xi(s-2k+1,\chi) = \left(\frac{q}{\pi}\right)^{s/2-k+1} \Gamma\left(\frac{s}{2} - k + 1\right) L(s-2k+1,\chi) = \Gamma^{\flat}_{k,q}(s) L(s-2k+1,\chi),$$

$$\xi(s+2k-1,\chi) = \left(\frac{q}{\pi}\right)^{s/2+k} \Gamma\left(\frac{s}{2}+k\right) L(s+2k-1,\chi)$$
$$= \Gamma^{\flat}_{k,q}(s)\left(\frac{q}{2\pi}\right)^{2k-1} (s-2k+2)\cdots(s-2)s(s+2)\cdots(s+2k-2)L(s+2k-1,\chi).$$

Applying Lemma 2.2, we have

$$\big|\xi(s+2k-1,\chi)\big| > \big|\xi(s-2k+1,\chi)\big|, \qquad \Re(s) > 1/2. \tag{2.8}$$

To avoid the poles of $\Gamma^{\flat}_{k,q}(s)$, we suppose that $s/2-k+1 \neq m$, where m is a non-positive integer. Dividing the both side hand of (2.8) by $|\Gamma^{\flat}_{k,q}(s)|$, we obtain

$$\begin{aligned}&\big|q^{2k-1}(s-2k+2)\cdots(s-2)s(s+2)\cdots(s+2k-2)L(s+2k-1,\chi)\big|\\ &> \big|(2\pi)^{2k-1}L(s-2k+1,\chi)\big|, \qquad \Re(s) > 1/2.\end{aligned}$$

When $\Re(s) > 1/2$, one has

$$\begin{aligned}&\big|(s)_{2k-1}\big| = \big|s(s+1)(s+2)(s+3)\cdots(s+2k-3)(s+2k-2)\big|\\ &= \big|(s+2k-3)(s+2k-5)\cdots(s+3)(s+1)s(s+2)\cdots(s+2k-4)(s+2k-2)\big|\\ &> \big|(s-2k+2)(s-2k+4)\cdots(s-4)(s-2)s(s+2)\cdots(s+2k-4)(s+2k-2)\big|\end{aligned}$$

according to

$$|s+1| > |s-2|, \quad |s+3| > |s-4|, \quad \ldots, \quad |s+2k-3| > |s-2k+2|.$$

Hence, by using $|q^{s+2k-1}| > |q^{2k-1}\sqrt{q}|$ with $\Re(s) > 1/2$, we obtain

$$\big|q^{s+2k}(s)_{2k-1}L(s+2k-1,\chi)\big| > \big|(2\pi)^{2k-1}\sqrt{q}L(s-2k+1,\chi)\big| \tag{2.9}$$

which implies that $R^{(2k-1)}(s,\chi)$ does not vanish if $\Re(s) > 1/2$ and $s/2-k+1 \neq m$.

Thus we only have to prove the case that $s/2-k+1 = m$, where m is a non-positive integer. The assumptions $s/2 = m+k-1$ and $\Re(s) > 1/2$ imply

$$1/4 < m+k-1 \leq k-1$$

which is equivalent to $5/2-2k < 2m \leq 0$. In this case, by $k \geq 1$, we have

$$2m+4k-3 > 2k-1/2 \geq 3/2, \qquad 2m-1 \leq -1.$$

Note that $L(2m-1,\chi) = 0$ when m is non-positive and χ is odd. Thus, we have

$$\begin{aligned}L(s+2k-1,\chi) &= L(2m+4k-3,\chi) > 0,\\ L(s-2k+1,\chi) &= L(2m-1,\chi) = 0\end{aligned} \tag{2.10}$$

when $s = 2m+2k-2 > 1/2$. We note that $R^{(2k-1)}(s,\chi)$ is expressed as

$$\begin{aligned}R^{(2k-1)}(s,\chi) &= R^{(2k-1)}_1(s,\chi) + R^{(2k-1)}_2(s,\chi),\\ R^{(2k-1)}_1(s,\chi) &:= (s)_{2k-1}q^{s+2k-1}L(s+2k-1,\chi),\\ R^{(2k-1)}_2(s,\chi) &:= (-1)^{k-1}(2\pi)^{2k-1}\sqrt{q}L(s-2k+1,\chi).\end{aligned}$$

Hence, by substituting $s = 2m+2k-2$ to $R^{(2k-1)}_1(s,\chi)$ and $R^{(2k-1)}_2(s,\chi)$, we have $R^{(2k-1)}(s,\chi) > 0$ if $\Re(s) > 1/2$ and $s/2-k+1 = m$. □

Proof of the Riemann hypothesis for $R^{(2k)}(s,\chi)$. Suppose $l=2k$ and χ is even, and put

$$\Gamma_{k,q}^{\sharp}(s) := \left(\frac{q}{\pi}\right)^{s/2+1/2-k} \Gamma\left(\frac{s}{2}-k\right).$$

From the definition of $\xi(s,\chi)$, one has

$$\xi(s-2k,\chi) = \left(\frac{q}{\pi}\right)^{s/2+1/2-k} \Gamma\left(\frac{s}{2}-k\right) L(s-2k,\chi) = \Gamma_{k,q}^{\sharp}(s) L(s-2k,\chi),$$

$$\begin{aligned}\xi(s+2k,\chi) &= \left(\frac{q}{\pi}\right)^{s/2+1/2+k} \Gamma\left(\frac{s}{2}+k\right) L(s+2k,\chi)\\ &= \Gamma_{k,q}^{\sharp}(s) \left(\frac{q}{2\pi}\right)^{2k} (s-2k)\cdots(s-2)s(s+2)\cdots(s+2k-2) L(s+2k,\chi),\end{aligned}$$

According to Lemma 2.2 again, we have

$$\bigl|\xi(s+2k,\chi)\bigr| > \bigl|\xi(s-2k,\chi)\bigr|, \qquad \Re(s) > 1/2. \tag{2.11}$$

Assume that $s/2-k \ne m$, where m is a non-positive integer to avoid the poles of $\Gamma_{k,q}^{\sharp}(s)$. Dividing the both side hand of (2.11) by $|\Gamma_{k,q}^{\sharp}(s)|$, we obtain

$$\bigl|q^{2k}(s-2k)\cdots(s-2)s(s+2)\cdots(s+2k-2)L(s+2k,\chi)\bigr| > \bigl|(2\pi)^{2k} L(s-2k,\chi)\bigr|.$$

In addition, when $\Re(s) > 1/2$, we have

$$\begin{aligned}\bigl|(s)_{2k}\bigr| &= \bigl|s(s+1)(s+2)(s+3)\cdots(s+2k-2)(s+2k-1)\bigr|\\ &= \bigl|(s+2k-1)(s+2k-3)\cdots(s+3)(s+1)s(s+2)\cdots(s+2k-4)(s+2k-2)\bigr|\\ &> \bigl|(s-2k)(s-2k+2)\cdots(s-4)(s-2)s(s+2)\cdots(s+2k-4)(s+2k-2)\bigr|\end{aligned} \tag{2.12}$$

by the inequalities

$$|s+2k-1| > |s-2k|, \quad |s+2k-3| > |s-2k+2|, \quad \ldots, \quad |s+1| > |s-2|.$$

Hence, from $|q^{s+2k}| > |q^{2k}\sqrt{q}|$, we have

$$\bigl|q^{s+2k}(s)_{2k} L(s+2k,\chi)\bigr| > \bigl|(2\pi)^{2k}\sqrt{q} L(s-2k,\chi)\bigr| \tag{2.13}$$

which implies $R^{(2k)}(s,\chi) \ne 0$ if $\Re(s) > 1/2$ and $s/2-k \ne m$.

Finally, suppose that $s/2-k=m$, where m is a non-positive integer. This assumption and the condition $\Re(s) > 1/2$ imply

$$1/4 < m+k \le k.$$

Then, from $k \ge 1$, we have

$$2m+4k > 2k+1/2 \ge 2, \qquad 2m \le 0.$$

It is well-known that $L(2m,\chi) = 0$ when m is non-positive and χ is even and non-primitive. Hence, one has

$$L(s+2k,\chi) = L(2m+4k,\chi) > 0, \qquad L(s-2k,\chi) = L(2m,\chi) = 0 \tag{2.14}$$

if $s = 2m+2k > 1/2$. Note that the function $R^{(2k)}(s,\chi)$ is written as

$$R^{(2k)}(s,\chi) = R_1^{(2k)}(s,\chi) + R_2^{(2k)}(s,\chi),$$

$$R_1^{(2k)}(s,\chi) := (s)_{2k} q^{s+2k} L(s+2k,\chi), \qquad R_2^{(2k)}(s,\chi) := (-1)^k (2\pi)^{2k} \sqrt{q} L(s-2k,\chi).$$

Thus, by substituting $s=2m+2k$ to $R_1^{(2k)}(s,\chi)$ and $R_2^{(2k)}(s,\chi)$, we obtain $R^{(2k)}(s,\chi)>0$ if $\Re(s)>1/2$ and $s/2-k=m$. □

Proof of the statements on the central values and real zeros of $R^{(l)}(s,\chi)$. Recall

$$R^{(l)}(s,\chi) = R_1^{(l)}(s,\chi) + R_2^{(l)}(s,\chi),$$
$$R_1^{(l)}(s,\chi) := (s)_l q^{s+l} L(s+l,\chi), \qquad R_2^{(l)}(s,\chi) := (2\pi)^l \psi(l)\sqrt{q} L(s-l,\chi).$$

Clearly, we have $R_1^{(2k-1)}(1/2,\chi)>0$ and $R_1^{(2k)}(1/2,\chi)>0$. According to (2.4), one has

$$L(1/2-4k+3,\chi)>0, \quad L(1/2-4k+1,\chi)<0 \qquad \chi \text{ is odd},$$
$$L(1/2-4k,\chi)>0, \quad L(1/2-4k+2,\chi)<0 \qquad \chi \text{ is even}.$$

Hence, we obtain $R_2^{(2k-1)}(1/2,\chi)>0$ and $R_2^{(2k)}(1/2,\chi)>0$. Therefore, the central value of $R^{(l)}(s,\chi)$ is positive.

By the assumption χ is primitive, the function $R^{(l)}(s,\chi)$ is entire. Thus, Riemann's functional equation and the fact $R^{(l)}(s,\chi)\ne 0$ for $\Re(s)>1/2$ imply that all real zeros of $R^{(l)}(s,\chi)$ are simple and located only at the non-positive even integers. □

2.3. **Proofs of Theorems 1.2 and 1.3.** To put it briefly, we show the functional equation and Riemann hypothesis for $\zeta_q^{(2k)}(s)$ by using (2.5) and Lemma 2.2, respectively. The Lindelöf hypothesis and (1.1) for $U(s,\chi)$ and $T_q(s)$ are proved by Lemma 2.1 and the Euler products of $\zeta(s)$ and $L(s,\chi)$.

Proof of Theorem 1.2. First, we show the following equations.

$$\sum_{r=1}^{q-1} Z(s,r/q) = 2\bigl(q^s-1\bigr)\zeta(s), \qquad \sum_{r=1}^{q-1} P(s,r/q) = 2\bigl(q^{1-s}-1\bigr)\zeta(s). \tag{2.15}$$

When $\sigma>1$, we have

$$\sum_{r=1}^{q-1}\zeta(s,r/q) = \sum_{r=1}^{q-1}\sum_{n=1}^{\infty}\frac{q^s}{(qn+r)^s} = q^s\zeta(s) - \sum_{n=1}^{\infty}\frac{q^s}{(qn+q)^s} = \bigl(q^s-1\bigr)\zeta(s),$$
$$\sum_{r=1}^{q-1}\mathrm{Li}_s(e^{2\pi i r/q}) = \sum_{r=1}^{q-1}\sum_{n=1}^{\infty}\frac{e^{2\pi i r/q}}{n^s} = q\sum_{n=1}^{\infty}\frac{1}{(qn)^s} - \sum_{n=1}^{\infty}\frac{1}{n^s} = \bigl(q^{1-s}-1\bigr)\zeta(s).$$

The equations above and the analytic continuation provide the formulas in (2.15) for all $s\in\mathbb{C}\setminus\{1\}$. From (2.7) and (2.15), we have

$$\begin{aligned}
\sum_{r=1}^{q-1} S^{(2k)}(s,r/q) &= \sum_{r=1}^{q-1} Z^{(2k)}(s,r/q) + \sum_{r=1}^{q-1} P^{(2k)}(s,r/q)\\
&= (-1)^{2k}(s)_{2k}\sum_{r=1}^{q-1} Z(s+2k,r/q) + (-1)^k(2\pi)^{2k}\sum_{r=1}^{q-1} P(s-2k,r/q)\\
&= (s)_{2k}\bigl(q^{s+2k}-1\bigr)\zeta(s+2k) + (-1)^k(2\pi)^{2k}\bigl(q^{1-s+2k}-1\bigr)\zeta(s-2k).
\end{aligned}$$

Thus, we obtain the functional equation $\zeta_q^{(2k)}(1-s) = \Gamma_{\cos}(s)\zeta_q^{(2k)}(s)$ by (2.5).

Second, we prove the Riemann hypothesis for $\zeta_q^{(2k)}(s)$. We define the Riemann xi-function $\xi(s)$ by
$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$
According to Lemma 2.2, we have
$$\bigl|\xi(s+2k)\bigr| > \bigl|\xi(s-2k)\bigr|, \qquad \Re(s) > 1/2. \tag{2.16}$$
Let $G_k(s) := \pi^{-s/2+k}\Gamma(s/2-k)$ and suppose $\Re(s) > 1/2$ and $s/2-k \ne m$, where m is a non-positive integer. From the definition of $\xi(s)$, we have
$$\begin{aligned} 2\xi(s-2k) &= (s-2k)(s-2k-1)\pi^{-s/2+k}\Gamma(s/2-k)\zeta(s-2k) \\ &= (s-2k)(s-2k-1)G_k(s)\zeta(s-2k), \end{aligned}$$
$$2\xi(s+2k) = (s+2k)(s+2k-1)\pi^{-s/2-k}\Gamma(s/2+k)\zeta(s+2k)$$
$$= (s+2k)(s+2k-1)G_k(s)(2\pi)^{-2k}(s-2k)\cdots(s-2)s(s+2)\cdots(s+2k-2)\zeta(s+2k).$$
Clearly, one has
$$\bigl|s+2k\bigr| > \bigl|s-2k-1\bigr|, \quad \bigl|s+2k-1\bigr| > \bigl|s-2k\bigr|, \qquad \Re(s) > 1/2.$$
The inequalities written above, (2.12) and (2.16) imply
$$\bigl|(s)_{2k}\zeta(s+2k)\bigr| > \bigl|(2\pi)^{2k}\zeta(s-2k)\bigr|, \qquad \Re(s) > 1/2, \tag{2.17}$$
if $s/2-k \ne m$, where m is a non-positive integer. We should note that $|G_k(s)| = \infty$ when $s/2-k = m$. Furthermore, we have
$$\bigl|q^{s+2k}-1\bigr| > \bigl|q^{1-s+2k}-1\bigr|, \qquad \Re(s) > 1/2. \tag{2.18}$$
This is proved by as follows. Let $C := \cos(t\log q)$. Then we have
$$\bigl|q^{s+2k}-1\bigr|^2 = 1+q^{4k+2\sigma}+2q^{2k+\sigma}C, \qquad \bigl|q^{1-s+2k}-1\bigr|^2 = 1+q^{4k+2-2\sigma}+2q^{2k+1-\sigma}C.$$
By an easy computation, we have
$$\bigl|q^{s+2k}-1\bigr|^2 - \bigl|q^{1-s+2k}-1\bigr|^2 = \bigl(q^\sigma - q^{1-\sigma}\bigr)\bigl(q^{4k+\sigma}+q^{4k+1-\sigma}+2q^{2k}C\bigr) > 0.$$
Therefore, for $\Re(s) > 1/2$ and $s/2-k \ne m$, we obtain
$$\bigl|(s)_{2k}\bigl(q^{s+2k}-1\bigr)\zeta(s+2k)\bigr| > \bigl|(2\pi)^{2k}\bigl(q^{1-s+2k}-1\bigr)\zeta(s-2k)\bigr|$$
by (2.17) and (2.18). Hence, $\zeta_q^{(2k)}(s) \ne 0$ when $\Re(s) > 1/2$ and $s/2-k \ne m$.

Next, suppose $\Re(s) > 1/2$ and $s/2-k = m$, where m is a non-positive integer. We can easily see that
$$\zeta(s+2k) = \zeta(2m+4k) > 0, \qquad \zeta(s-2k) = \zeta(2m) = 0.$$
if $m \le -1$. Hence, $\zeta_q^{(2k)}(s) > 0$ when $s/2-k = m \le -1$. Now assume that $s/2-k = 0$, namely, $s = 2k$. From $\zeta(4k) > 1$ and $\zeta(0) = -1/2$, we have
$$\begin{aligned} \zeta_q^{(2k)}(2k) &= (2k)_{2k}\bigl(q^{4k}-1\bigr)\zeta(4k) + (-1)^k(2\pi)^{2k}(q-1)\zeta(0) \\ &> (2k)_{2k}\bigl(q^{4k}-1\bigr) - (2\pi)^{2k}(q-1) > (2k)^{2k}\bigl(q^{4k}-1\bigr) - (2\pi)^{2k}(q-1) \\ &> (2k)^{2k}q^{2k+1} - (2\pi)^{2k}q = q\bigl((2kq)^{2k} - (2\pi)^{2k}\bigr) > 0 \end{aligned}$$
when $k \ge 2$. If $k = 1$, we have
$$\zeta_q^{(2)}(2) = (2)_2\bigl(q^4-1\bigr)\zeta(4) - (2\pi)^2(q-1)\zeta(0) > 6\bigl(q^4-1\bigr)\zeta(4) > 0$$

by $\zeta(0) = -1/2$ again. Therefore, $\zeta_q^{(2k)}(s) > 0$ when $s/2 - k = m$, where m is a non-positive integer.

Finally, we prove the statements on the central values and real zeros of $\zeta_q^{(2k)}(s)$. We obviously have $\zeta(1/2 + 2k) > 0$. From the functional equation (1.4), one has

$$\zeta(1/2 - 4k) > 0, \qquad \zeta(1/2 - 4k + 2) < 0.$$

Therefore, we have $\zeta_q^{(2k)}(1/2) > 0$. By the definition, the function $\zeta_q^{(2k)}(s)$ is entire. Hence, Riemann's functional equation and the fact $\zeta_q^{(2k)}(s) \neq 0$ for $\Re(s) > 1/2$ imply that all real zeros of $\zeta_q^{(2k)}(s)$ are simple and located only at the non-positive even integers. □

Proof of Theorem 1.3. Since the functions $R^{(2)}(s,\chi)$ and $\zeta_q^{(2)}(s)$ are entire and have real simple zeros only on non-positive even integers, we can see that $U(s,\chi)$ and $T_q(s)$ have a pole at $s = 1$ and real zeros at only negative even integers. Hence, we show the Lindelöf hypothesis for $U(s,\chi)$ and $T_q(s)$. Applying Lemma 2.1, we have

$$U(s,\chi),\ T_q(s) = O_{q,\sigma}(1), \quad \sigma \geq 1/2.$$

Thus, by functional equations for $U(s,\chi)$ and $T_q(s)$ and Stirling's approximation

$$\bigl|\Gamma(s)\bigr| = \sqrt{2\pi} e^{-\pi|t|/2} |t|^{\sigma-1/2} \bigl(1 + O(|t|^{-1})\bigr) \tag{2.19}$$

(see [11, (1.5.7)] or [19, (4.12.2) and (4.12.3)]), we obtain

$$U(s,\chi),\ T_q(s) \ll_{q,\sigma} |t|^{\mu(\sigma)},$$

where $\mu(\sigma)$ is given in (1.2). From the Euler products of $L(s,\chi)$ and $\zeta(s)$, we have

$$\frac{(s)_2 L(s+2,\chi)}{s(1-s)},\ \frac{(s)_2 \zeta(s+2)}{s(1-s)} \gg_{q,\sigma} \prod_p \frac{1}{1+p^{\sigma+2}} \gg_{q,\sigma} 1$$

when $\sigma, |t| \geq 1$. On the other hand, by using the estimation

$$L(s,\chi),\ \zeta(s) \ll_{q,\sigma} \begin{cases} |t|^{1/2-\sigma} & \sigma \leq 0, \\ |t|^1 & \sigma > 0, \end{cases}$$

we can see that

$$\frac{L(s-2,\chi)}{s(1-s)},\ \frac{\zeta(s-2)}{s(1-s)} \ll_{q,\sigma} \begin{cases} |t|^{-2}|t|^{5/2-\sigma} & -1 \leq \sigma - 2 \leq 0 \\ |t|^{-2}|t|^1 & \sigma - 2 > 0 \end{cases} \ll_{q,\sigma} |t|^{-1/2}$$

for $\sigma \geq 1$. Hence, when $\sigma \geq 1$, we obtain

$$U(s,\chi),\ T_q(s) \gg_{q,\sigma} 1.$$

The estimation above and Riemann's functional equations for $U(s,\chi)$ and $T_q(s)$ imply the estimation (1.6). □

2.4. **Proofs of Theorem 1.5, Propositions 1.4 and 1.6.** In this subsection, we show results on $R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b})$ defined as (1.7).

Proof of Proposition 1.4. First we define three functions as

$$\xi(s,\boldsymbol{\chi},\boldsymbol{b}) := \left(\frac{q}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{s+\kappa(\chi)}{2}\right) \sum_{h=1}^{j} b_h L(s,\chi_h),$$

$$R_1^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b}) := \sum_{h=1}^{j} b_h (s)_l q^{s+l} L(s+l,\chi_h), \tag{2.20}$$

$$R_2^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b}) := \sum_{h=1}^{j} b_h (2\pi)^l \psi(l) \sqrt{q} L(s-l,\chi_h).$$

The functional equation (1.8) is easily proved by (2.5) by the assumption that all $\chi_1,\ldots,\chi_j$ are odd or even. From the Dirichlet series expression of $L(s,\chi)$, one has

$$\frac{1}{q^{\sigma+l}(\sigma)_l} R^{(l)}(\sigma,\boldsymbol{\chi},\boldsymbol{b}) = \sum_{h=1}^{j} b_h + o(1), \qquad \sigma \to +\infty.$$

Thus, there exits $\sigma_0 > 1$ such that the function $R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b})$ does not vanish for all $\Re(s) > \sigma_0$ from the assumption $b_1,\ldots,b_j > 0$. Hence, there is $\alpha_0 > 0$ such that all zeros of $\sum_{h=1}^{j} b_h L(s,\chi_h)$ lie in the vertical strip $|\Re(s) - 1/2| < \alpha_0$. Now suppose that $\chi_1,\ldots,\chi_j$ are odd. By using Proposition 2.2 we can see that there exist $l_0 \in \mathbb{N}$ such that for any $2l-1 \ge l_0$,

$$\bigl|\xi(s+2l-1,\boldsymbol{\chi},\boldsymbol{b})\bigr| > \bigl|\xi(s-2l-1,\boldsymbol{\chi},\boldsymbol{b})\bigr|.$$

When $s/2 - l + 1 \ne m$, where m is a non-positive integer, we have

$$\bigl|R_1^{(2l-1)}(s,\boldsymbol{\chi},\boldsymbol{b})\bigr| > \bigl|R_2^{(2l-1)}(s,\boldsymbol{\chi},\boldsymbol{b})\bigr|.$$

by modifying the proof of (2.9). If $s/2-l+1 \ne m$, we can show the inequality above form (2.10). When $s/2-l+1 = m$, we have $R^{(2l-1)}(s,\boldsymbol{\chi},\boldsymbol{b}) \ne 0$ by (2.10) and the assumption $b_1,\ldots,b_j > 0$. Therefore, functions $R^{(2l-1)}(s,\boldsymbol{\chi},\boldsymbol{b})$ does not vanish when $\Re(s) > 1/2$. We can prove that $R^{(2l)}(s,\boldsymbol{\chi},\boldsymbol{b}) \ne 0$ for $\Re(s) > 1/2$ by modifying the proof of (2.13) and using (2.14). $\square$

Proof of Thereom 1.5. Riemann's functional equation and the Lindelöf hypothesis for the function $U(s,\boldsymbol{\chi},\boldsymbol{b})$ are prove by the argument in the proofs of Proposition 1.4 and Theorem 1.3, respectively. Hence, we show that $U(s,\boldsymbol{\chi},\boldsymbol{b})$ does not satisfy an analogue of the Riemann hypothesis for some $b_1,\ldots,b_j \in \mathbb{C}\setminus\{0\}$. Let $j=2$ and fix $s_0 \in \mathbb{C}$ satisfying $\Re(s_0) > 1/2$. Then we define $0 \ne c_0 \in \mathbb{C}$ by

$$c_0 := \frac{R^{(2)}(s_0,\chi_1)}{R^{(2)}(s_0,\chi_2)},$$

where χ_1 and χ_2 are different real primitive Dirichlet characters. Note that $R^{(2)}(s_0,\chi_2)$ and $R^{(2)}(s_0,\chi_2)$ are not zero if $\Re(s_0) > 1/2$ by Theorem 1.1. Then, obviously, the function

$$\frac{R^{(2)}(s,\chi_1)}{s(1-s)} - c_0 \frac{R^{(2)}(s,\chi_2)}{s(1-s)}$$

has a zero at $s = s_0$ from the definitions of $c_0 \in \mathbb{C}$ and $s_0 \in \mathbb{C}$. $\square$

Proof of Proposition 1.6. Let $R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b}) = (s)_l q^{s+l} R^{(l)}_*(s,\boldsymbol{\chi},\boldsymbol{b})$. Recall that there exits $\sigma_0 > 1$ such that the function $R^{(l)}(s,\boldsymbol{a},\boldsymbol{b})$ does not vanish for all $\Re(s) > \sigma_0$ (see the proof of Proposition 1.4). From the argument in the proof of [19, Theorem 9.3], functional equations (1.4) and (1.8), we have

$$\pi N(T,R_1) = \Delta \arg \pi^{-s/2} + \Delta \arg \Gamma(s/2) + \Delta \arg (s)_l q^{s+l} + \Delta \arg R^{(l)}_*(s,\boldsymbol{\chi},\boldsymbol{b}),$$

where Δ denotes the variation from σ_0 to $\sigma_0 + iT$, and then to $1/2 + iT$, along straight lines. By the estimations in the proof of [19, Theorem 9.3], we obtain

$$\begin{aligned}&\Delta \arg \pi^{-s/2} + \Delta \arg \Gamma(s/2) + \Delta \arg (s)_l q^{s+l}\\ =&\frac{T}{2}\log\frac{T}{2} - \frac{T}{2} - \frac{T}{2}\log \pi + T\log q + O(1).\end{aligned}$$

Now we consider $\Delta \arg R^*_1(s)$. Clearly, there exists $\sigma_1 \ge \sigma_0$ and m_1 such that

$$\left|\Re\bigl(R^{(l)}_*(\sigma_1,\boldsymbol{\chi},\boldsymbol{b})\bigr)\right| > m_1.$$

Applying Proposition 2.3 with $f(s) = R^*_1(s)$, $\alpha = 0$ and $\beta = 1/2$, we obtain

$$\Delta \arg R^{(l)}_*(s,\boldsymbol{\chi},\boldsymbol{b}) = O(\log T)$$

by Lemma 2.1. Therefore, we have (1.9). □

3. Remarks

3.1. **Lindelöf hypothesis.** Constant functions satisfy the LH but do not fulfill (1.1). Taylor's function $\zeta^*(s+1/2) - \zeta^*(s-1/2)$ mentioned in Section 1.1 does not satisfy (1.1) by Stirling's approximation (2.19). However, his function fulfills both the LH and RH.

Under the LH of Dirichlet L-functions, the function $\sum_{h=1}^{j} b_h L(s,\chi_h)$ satisfies the LH. From [17, Theorem], the function $\sum_{h=1}^{j} b_h L(s,\chi_h)$ has infinitely many zeros in both the vertical strip $1/2 < \Re(s) < 1$ and the half-plane $\Re(s) > 1$ if $j \ge 2$ and $b_h \ne 0$ for all $1 \le h \le j$. Let $j \ge 2$ and $b_h \ne 0$ and all Dirichlet characters be even (or odd) $\bmod q$. In this case, $\sum_{h=1}^{j} b_h L(s,\chi_h)$ does not satisfy (1.1) by the zeros in the half-plane $\Re(s) > 1$.

Therefore, it is difficult to find functions which satisfy the condition (1.1). Furthermore, it should be emphasised that $U(s,\chi)$ and $T_q(s)$ have a simple pole at $s = 1$ and simple real zeros only at the negative even integers just like $\zeta(s)$.

3.2. **Infinite product representation.** Recall the functions $R^{(l)}_1(s,\boldsymbol{\chi},\boldsymbol{b})$ and $R^{(l)}_2(s,\boldsymbol{\chi},\boldsymbol{b})$ are given in (2.20). Clearly one has

$$R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b}) = R^{(l)}_1(s,\boldsymbol{\chi},\boldsymbol{b}) + R^{(l)}_2(s,\boldsymbol{\chi},\boldsymbol{b}).$$

Suppose $|R^{(l)}_1(s,\boldsymbol{\chi},\boldsymbol{b})| > |R^{(l)}_2(s,\boldsymbol{\chi},\boldsymbol{b})|$ when $\Re(s) > 1/2$ and does not vanish identically. Then we have

$$R^{(l)}(s,\boldsymbol{\chi},\boldsymbol{b}) = R^{(l)}_1(s,\boldsymbol{\chi},\boldsymbol{b}) \exp\left(\sum_{m=1}^{\infty} N_m \frac{u(s,\boldsymbol{\chi},\boldsymbol{b})^m}{m}\right), \tag{3.1}$$

where N_m and $u(s,\boldsymbol{\chi},\boldsymbol{b})$ are defined as

$$N_m := (-1)^{m+1}, \qquad u(s,\boldsymbol{\chi},\boldsymbol{b}) = \frac{R^{(l)}_2(s,\boldsymbol{\chi},\boldsymbol{b})}{R^{(l)}_1(s,\boldsymbol{\chi},\boldsymbol{b})}.$$

Noted that the function above looks like the local zeta function or the congruent zeta function. We can prove the infinite product representation (3.1) by modifying the proof in [16, Section 3.1].

3.3. **Hardy's $\boldsymbol{Z}$-function.** Let

$$\eta(s) := \frac{1}{\Gamma_{\cos}(s)} = \frac{\Gamma(1/2 - s/2)}{\Gamma(s/2)}\pi^{s-1/2}.$$

By using $\eta(s)$ above, we define Hardy's Z-function $Z(t)$ by

$$Z(t) := \bigl(\eta(1/2+it)\bigr)^{-1/2}\zeta(1/2+it) = e^{i\theta(t)}\zeta(1/2+it),$$

where $\theta(t) := \Im(\log\Gamma(1/4+it/2)) - (t/2)\log\pi$. It is well known (e.g. [5, Chapter 1.3]) that for $t \in \mathbb{R}$,

$$Z(t) \in \mathbb{R}, \qquad |Z(t)| = |\zeta(1/2+it)|, \qquad Z(t) = Z(-t).$$

Since $R^{(l)}(s, \boldsymbol{\chi}, \boldsymbol{b})$ is real on the real line and satisfy Riemann's functional equation, we can define

$$H^{(l)}(t, \boldsymbol{\chi}, \boldsymbol{b}) := e^{i\theta(t)} R^{(l)}(1/2+it, \boldsymbol{\chi}, \boldsymbol{b}), \qquad j = 1, 2, 3$$

as an analogue of $Z(t)$. By modifying the argument in [5, Chapter 1.3], we have

$$H^{(l)}(t, \boldsymbol{\chi}, \boldsymbol{b}) \in \mathbb{R}, \quad \bigl|H^{(l)}(t, \boldsymbol{\chi}, \boldsymbol{b})\bigr| = \bigl|R^{(l)}(1/2+it, \boldsymbol{\chi}, \boldsymbol{b})\bigr|, \quad H^{(l)}(t, \boldsymbol{\chi}, \boldsymbol{b}) = H^{(l)}(-t, \boldsymbol{\chi}, \boldsymbol{b}).$$

Note that the cases when $l = 1$ and $q = 3, 4$ have already treated in [16, Section 3.2].

3.4. **Numerical calculation.** Recall that $T_q(s)$ and $U(s, \chi)$ given in Theorem 1.3 satisfy Riemann's functional equation and the Lindelöf and Riemann hypotheses. We define two functions $H_1(t)$ and $H_2(t)$ by

$$H_1(t) := e^{i\theta(t)} T_2(1/2+it), \qquad H_2(t) := e^{i\theta(t)} U(1/2+it, \chi_5),$$

where χ_5 is the real non-primitive Dirichlet character mod 5. The following figures are given by Mathematica 13.0. It should be noted that they are plotted by not $H_j(t)$ but $\Re(H_j(t))$ because Mathematica 13.0 can not regard $H_j(t)$ as real functions*.

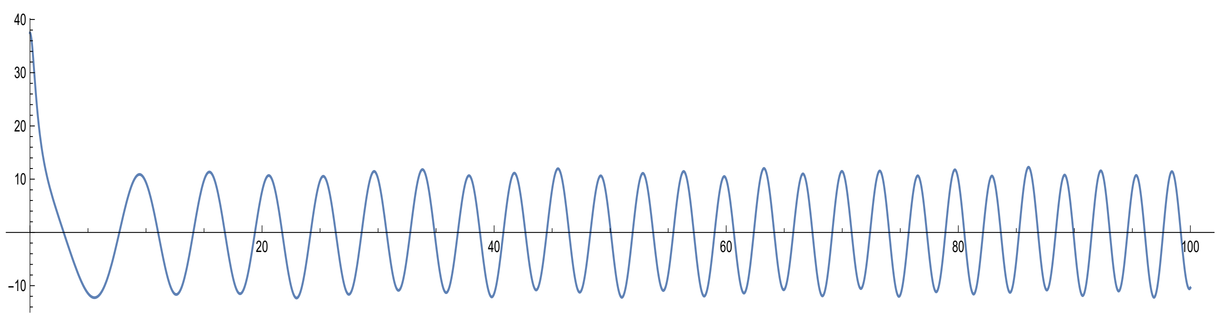

FIGURE 1. $\{H_1(t) : 0 \le t \le 100\}$

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*In [16, Section 3.3], all $H_j(1/2+it)$ should be replaced by $H_j(t)$.

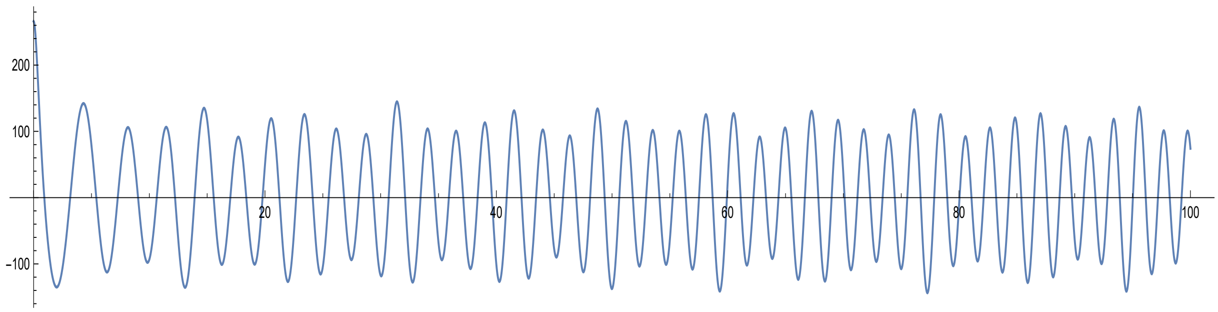

FIGURE 2. $\{H_2(t) : 0 \leq t \leq 100\}$

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