Analysis of the Moon's Orbit Using Perturbation Theory: Orbital Changes Owing to Solar Gravity in the Earth-Moon Two-Body System and Its Applications in Physics Education.

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Abstract

This study analyzes the Moon's orbit by considering the perturbation effects of the Sun within the Earth-Moon system. A temporal variation in the Moon's orbital eccentricity under the influence of the Sun's tidal force was revealed, with notable changes observable within a six-month cycle. This variation elucidates widely known phenomena such as the changing visual diameters of supermoons. While our results have significant implications for understanding perturbation theory in physics education, the discrepancies between our model and the observed geocentric distances suggest the need to address the limitations of point-mass approximations. Through perturbation theory, this study underscores the importance of qualitative insights in physics education, particularly regarding celestial motions, and highlights areas for future modeling refinement.

Keywords: the Moon's orbit, perturbation, tidal force, three-body problem

1 Introduction

Perturbation theory comprises methods for obtaining approximate solutions to phenomena in physics where exact solutions cannot be obtained. In other words, perturbation theory offers a qualitative understanding of motion and cultivates physical intuition without providing accurate insights revealed by the exact solutions.

As the importance of perturbation theory is widely acknowledged, the lack of adequate emphasis on applying this theory to natural phenomena in the general post-secondary physics education curriculum is problematic. Models that closely reflect reality can capture students' interests and enable a profound understanding of natural phenomena. Therefore, we propose a method to enhance the understanding of perturbation theory in physics education by applying it to the Moon's orbit, which is a familiar natural phenomenon [1].

First, the Earth, which is affected only by the gravity of the Sun, is known to trace an elliptical path with the Sun at its focus (see Section A.3). Similarly, the Moon's orbit can be approximated as an ellipse with the Earth at its focus. However, in reality, multiple celestial bodies exert gravity on the Moon to alter its orbit to a shape different from an ellipse. This is also why the apparent diameter of supermoons

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[1] Analysis conducted as the following are known: the Moon's orbital elements in the Earth-Moon system, the Earth's orbital elements in the Sun-Earth binary system, and the mass of the Sun, the Earth, and the Moon[1].

varies. We attempt to approximate the Moon's orbital change using perturbation theory, considering the effect of the Sun's gravity as a perturbation to the Moon's orbit.

Newton's equations of motion (EoMs) are complex in the Earth-Moon-Sun system. Therefore, we solve the Euler-Lagrange equation to derive the equation of the Moon's orbital motion, revealing that the Sun's gravity can be considered a perturbation to the Earth-Moon system.

The Moon's orbit in the Earth-Moon system is determined by its semi-major axis and eccentricity, which are known to depend on the Moon's angular momentum and mechanical energy (see Section A.3). Owing to the Sun's perturbation, the Moon's angular momentum and mechanical energy vary with time. We derive these time-dependent factors to determine the Moon's orbit.

Our analysis shows that the Moon's orbital eccentricity varies over time and that the local minima of the geocentric distance exhibit variations within a six-month period. Additionally, we demonstrate that the Moon's orbit in the Earth-Moon system is modified by the tidal force exerted by the Sun. These results demonstrate how a tidal force, a phenomenon highly relevant to our daily lives, can be used to elucidate perturbation theory and provide valuable insights for physical education.

2 Methods

2.1 Lagrangian L in the Earth-Moon-Sun system

In this section, we define the Lagrangian L in the Earth-Moon-Sun system to derive the Moon's EoM with respect to the Earth. Let M_{S}, M_{E}, and M_{M} represent the masses of the Sun, Earth, and Moon, respectively. We denote the positions of the Earth and Moon in an absolute rest frame with the Sun as the origin as $\boldsymbol{r}_{\mathrm{E}}$ and $\boldsymbol{r}_{\mathrm{M}}$, respectively. L in this system is given as follows:

$$L = \frac{1}{2}M_{\mathrm{E}}\dot{\boldsymbol{r}}_{\mathrm{E}}^2 + \frac{1}{2}M_{\mathrm{M}}\dot{\boldsymbol{r}}_{\mathrm{M}}^2 + G\left(\frac{M_{\mathrm{S}}M_{\mathrm{E}}}{|\boldsymbol{r}_{\mathrm{E}}|} + \frac{M_{\mathrm{S}}M_{\mathrm{M}}}{|\boldsymbol{r}_{\mathrm{M}}|} + \frac{M_{\mathrm{E}}M_{\mathrm{M}}}{|\boldsymbol{r}_{\mathrm{E}} - \boldsymbol{r}_{\mathrm{M}}|}\right), \tag{1}$$

where G denotes the universal gravitational constant.

We further rewrite Equation (1) using several new parameters: the position of the Moon with respect to the Earth $\boldsymbol{r}$, the center of gravity of the Earth and Moon $\boldsymbol{r}_{\mathrm{G}}$, the reduced mass of the Earth and Moon μ, and the sum of the masses of the Earth and Moon M_{G}.

By performing a Maclaurin expansion of the potential terms in Equation (1) and retaining the terms up to the second order in $|\boldsymbol{r}|/|\boldsymbol{r}_{\mathrm{G}}|$, we obtain

$$L = \frac{1}{2}M_{\mathrm{G}}\dot{\boldsymbol{r}}_{\mathrm{G}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 + G\left(\frac{M_{\mathrm{S}}M_{\mathrm{G}}}{r_{\mathrm{G}}} + \frac{\mu M_{\mathrm{G}}}{r} - \frac{\mu M_{\mathrm{S}}}{2}\frac{\boldsymbol{r}^2}{r_{\mathrm{G}}^3} + \frac{3}{2}\mu M_{\mathrm{S}}\frac{(\boldsymbol{r}\cdot\boldsymbol{r}_{\mathrm{G}})^2}{r_{\mathrm{G}}^5}\right), \tag{2}$$

where $r = |\boldsymbol{r}|$, $r_{\mathrm{G}} = |\boldsymbol{r}_{\mathrm{G}}|$, and $|\boldsymbol{r}|/|\boldsymbol{r}_G| \sim 10^{-3}$. Note that the third and fourth potential terms correspond to the Sun's primary and secondary small perturbation terms, respectively.

2.2 The Equations of Motion of the Moon

This section demonstrates how the Sun's gravity can be perceived as a perturbation to the Earth-Moon system, deriving the Moon's EoM in the Earth-Moon-Sun three-body system.

Using the Euler-Lagrange equation and Equation (2), we express the EoM with respect to $\boldsymbol{r}$ as follows:

$$\mu\ddot{\boldsymbol{r}} = -G\mu M_{\mathrm{G}}\left(\frac{\boldsymbol{r}}{r^3}\right) - G\mu M_{\mathrm{S}}\left(\frac{\boldsymbol{r}}{r_{\mathrm{G}}^3}\right) + 3G\mu M_{\mathrm{S}}\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_{\mathrm{G}}}{r_{\mathrm{G}}^5}\right)\boldsymbol{r}_{\mathrm{G}}. \tag{3}$$

We set the first, second, and third terms on the right-hand side of Equation (3) as $\boldsymbol{F}_0$, $\boldsymbol{F}_1$, and $\boldsymbol{F}_2$, respectively. $\boldsymbol{F}_0$ represents the gravitational interaction between the Earth and the Moon, while the terms $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ represent that between the Sun and the Moon.

Subsequently, we compare the sizes of $\boldsymbol{F}_0$, $\boldsymbol{F}_1$, and $\boldsymbol{F}_2$. By setting a_0 as the Moon's semi-major axis in the Earth-Moon system, A_0 as the Earth's semi-major axis in the Sun-Earth binary system, and noting that $|\boldsymbol{r}| \simeq a_0$ and $|\boldsymbol{r}_\mathrm{G}| \simeq A_0$, we can evaluate the following: [1]:

$$\frac{F_1}{F_0} \sim \frac{F_2}{F_0} \sim 10^{-3}. \tag{4}$$

Based on the above equation, it is evident that the effect of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ on the Moon's motion is significantly smaller than that of $\boldsymbol{F}_0$, indicating that the effect of the Sun's gravity, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, can be considered as perturbation. Furthermore, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are the first- and second-order small perturbation terms of the Sun, respectively.

Equation (4) suggests that it is necessary to analyze up to the second order because it considers the perturbation of the Sun on the Earth-Moon system.[2]

2.3 The Moon's Orbit in the Earth-Moon-Sun System

In the Earth-Moon system, the Moon's orbit is determined by its orbital semi-major axis a_0 and eccentricity ϵ_0, which are both constant. a_0 and ϵ_0 are determined by the Moon's angular momentum $\boldsymbol{l_0}$ and mechanical energy E_0, which are also constant (see Section A.1,A.2).

Considering the perturbation of the Sun, this study assumes that $\boldsymbol{l_0}$ and E_0 are non-conserved and time-dependent quantities. Using the relationship of the Earth-Moon system, the Moon's orbit in the Earth-Moon-Sun system $r(t)$ is expressed as follows:

$$r(t) = a(t)\left(1 - \epsilon(t)\cos\psi_\mathrm{M}(t)\right), \tag{5}$$

where $\psi_\mathrm{M}(t)$ is the anomaly of the Moon. Because $a(t)$ and $\epsilon(t)$ are the Moon's orbital semi-major axis and eccentricity, respectively,

$$a(t) = -\frac{G\mu M_\mathrm{G}}{2E(t)}, \tag{6}$$

$$\epsilon(t) = \sqrt{1 + \frac{2l^2(t)E(t)}{G^2\mu^2 M_\mathrm{G}^2}}, \tag{7}$$

where $l_0 = |\boldsymbol{l_0}|$. To obtain $r(t)$, we derive the equations for the Moon's angular momentum $|\boldsymbol{l}(t)|$ and mechanical energy $E(t)$ using the equation of motion (3).

2.4 The time evolution of the Moon's angular momentum

Considering the perturbation caused by the Sun in the Earth-Moon system, we derive the Moon's angular momentum $\boldsymbol{l}(t)$.

Taking the cross product of Equation (3) and $\boldsymbol{r}$, we derive the time derivative of $\boldsymbol{l}(t) = \mu(\boldsymbol{r} \times \dot{\boldsymbol{r}})$ as follows:

$$\frac{d}{dt}\boldsymbol{l}(t) = 3G\mu M_\mathrm{S}\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_\mathrm{G}}{|\boldsymbol{r}_\mathrm{G}|^5}\right)(\boldsymbol{r} \times \boldsymbol{r}_\mathrm{G}). \tag{8}$$

If we assume the Moon's angular momentum in the Earth-Moon system to be $\boldsymbol{l}_0$ and the correction of the Moon's angular momentum caused by the perturbation to be $\Delta\boldsymbol{l}(t)$, then

$$\boldsymbol{l}(t) = \boldsymbol{l}_0 + \Delta\boldsymbol{l}(t). \tag{9}$$

Therefore, using Equations (8) and (9), the time derivative of $\Delta\boldsymbol{l}(t)$ becomes

$$\frac{d}{dt}\Delta\boldsymbol{l}(t) = 3G\mu M_\mathrm{S}\left(\frac{\boldsymbol{r}\cdot\boldsymbol{r}_\mathrm{G}}{|\boldsymbol{r}_\mathrm{G}|^5}\right)(\boldsymbol{r} \times \boldsymbol{r}_\mathrm{G}), \tag{10}$$

[2] The details in the section2.5

$\boldsymbol{r}$ and $\boldsymbol{r}_{\mathrm{G}}$ can be expressed as follows:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \Delta\boldsymbol{r}, \tag{11}$$

$$\boldsymbol{r}_{\mathrm{G}} = \boldsymbol{r}_{\mathrm{G}0} + \Delta\boldsymbol{r}_{\mathrm{G}}, \tag{12}$$

where $\boldsymbol{r}_0$ is the position of the Moon with respect to the Earth in the Earth-Moon system; $\boldsymbol{r}_{\mathrm{G}0}$ is the center of mass of the Moon and Earth with respect to the Sun; $\Delta\boldsymbol{r}$ is the perturbative correction of the Moon's position in the Earth-Moon system, and $\Delta\boldsymbol{r}_{\mathrm{G}0}$ is the perturbative correction of the center of mass of the Moon and Earth from the Sun.

If we perform a Taylor expansion of Equation (10) around $\boldsymbol{r}_0$ and $\boldsymbol{r}_{\mathrm{G}0}$ for $\boldsymbol{r}$ and $\boldsymbol{r}_{\mathrm{G}}$, respectively, the most significant terms can be extracted as follows:

$$\frac{d}{dt}\Delta\boldsymbol{l}(t) = 3G\mu M_{\mathrm{S}}\left(\frac{\boldsymbol{r}_0\cdot\boldsymbol{r}_{\mathrm{G}0}}{|\boldsymbol{r}_{\mathrm{G}0}|^5}\right)(\boldsymbol{r}_0\times\boldsymbol{r}_{\mathrm{G}0}). \tag{13}$$

Assuming that the orbital plane of the Moon lies in the $x-y$ plane and that the angle between the orbital plane of the Moon and that of the Earth is α, we can derive

$$\boldsymbol{r}_0 = \begin{pmatrix} r_0(t)\cos\psi_{\mathrm{M}}(t) \\ r_0(t)\sin\psi_{\mathrm{M}}(t) \\ 0 \end{pmatrix}, \tag{14}$$

$$\boldsymbol{r}_{\mathrm{G}0} = \begin{pmatrix} r_{\mathrm{G}0}(t)\cos\psi_{\mathrm{E}}(t) \\ r_{\mathrm{G}0}(t)\sin\psi_{\mathrm{E}}(t)\cos\alpha \\ -r_{\mathrm{G}0}(t)\sin\psi_{\mathrm{E}}(t)\sin\alpha \end{pmatrix}, \tag{15}$$

where $r_0(t) = |\boldsymbol{r}_0|$ and $r_{\mathrm{G}0}(t) = |\boldsymbol{r}_{\mathrm{G}0}|$. $\psi_{\mathrm{E}}(t)$ is the Earth anomaly in the Earth-Moon system. Additionally, $r_0(t)$ and $r_{\mathrm{G}0}(t)$ satisfy

$$r_0(t) = a_0(1-\epsilon_0\cos\psi_{\mathrm{M}}(t)), \tag{16}$$

$$r_{\mathrm{G}0}(t) = A_0(1-\epsilon_{\mathrm{E}0}\cos\psi_{\mathrm{E}}(t)), \tag{17}$$

where $\epsilon_{\mathrm{E}0}$ is the orbital eccentricity of the Earth in the Sun-Earth binary system. Based on the difference between the anomalies of the Moon and Earth $\varphi(t) = \psi_{\mathrm{M}}(t) - \psi_{\mathrm{E}}(t)$, Equations (13), (14), (15), and (17) the time derivative of $\Delta l(t)$ is

$$\begin{aligned}\frac{2}{3G\mu M_{\mathrm{S}}}\frac{A_0^3}{a_0^2}\frac{d}{dt}\Delta l(t) = \sin 2\varphi(t) - \epsilon_0\Big(\sin(3\psi_{\mathrm{M}}(t)-2\psi_{\mathrm{E}}) + \sin(\psi_{\mathrm{M}}-2\psi_{\mathrm{E}})\Big)\\ +\frac{3}{2}\epsilon_{\mathrm{E}0}\Big(\sin(2\psi_{\mathrm{M}}-\psi_{\mathrm{E}}) + \sin(2\psi_{\mathrm{M}}-3\psi_{\mathrm{E}})\Big),\end{aligned} \tag{18}$$

In light of the established values ϵ_0 and $\epsilon_{\mathrm{E}0}$ being significantly smaller than 1 [1], we consider the terms up to the first order of ϵ_0 and $\epsilon_{\mathrm{E}0}$. Integrating this, $\Delta l(t)$ becomes

$$\Delta l(t) = \frac{3}{2}G\mu M_{\mathrm{S}}\frac{a_0^2}{A_0^3}C_l(t), \tag{19}$$

where we can define

$$\begin{aligned}C_l(t) \equiv -\frac{\cos 2\varphi}{2\dot\varphi} + \epsilon_0\left(\frac{\cos(3\psi_{\mathrm{M}}-2\psi_{\mathrm{E}})}{3\dot\psi_{\mathrm{M}}-2\dot\psi_{\mathrm{E}}} + \frac{\cos(\psi_{\mathrm{M}}-2\psi_{\mathrm{E}})}{\dot\psi_{\mathrm{M}}-2\dot\psi_{\mathrm{E}}}\right)\\ -\frac{3}{2}\epsilon_{\mathrm{E}0}\left(\frac{\cos(2\psi_{\mathrm{M}}-\psi_{\mathrm{E}})}{2\dot\psi_{\mathrm{M}}-\dot\psi_{\mathrm{E}}} + \frac{\cos(2\psi_{\mathrm{M}}-3\psi_{\mathrm{E}})}{2\dot\psi_{\mathrm{M}}-3\dot\psi_{\mathrm{E}}}\right),\end{aligned} \tag{20}$$

where $\sin\alpha \ll 1$ and $\epsilon_0 \sim \epsilon_{\mathrm{E0}} \ll 1$ are established [1], we consider the terms up to the first order of $\sin\alpha$, ϵ_0, and ϵ_{E0}. The integral constant in Equation (19) can be set to zero without loss of generality because we investigate the periodicity of the Moon's orbit owing to the perturbation by the Sun.

We further derive $l(t)$ with l_0. In the Earth-Moon system, l_0 is represented as follows:

$$l_0 = \mu\sqrt{a_0 G M_{\mathrm{G}}}. \tag{21}$$

Therefore, by referring to Equations (9), (19), and (21), $l(t)$ is obtained as follows:

$$l(t) = l_0\left(1 + \frac{3}{2}\nu C_l(t)\right), \tag{22}$$

where

$$\nu \equiv \sqrt{\frac{a_0 G}{M_{\mathrm{G}}}} M_{\mathrm{S}} \frac{a_0}{A_0^3}. \tag{23}$$

2.5 Time Evolution of the Mechanical Energy of the Moon

In this section, we derive the Moon's mechanical energy $E(t)$ from the perturbation by the Sun in the Earth-Moon system.

Taking the dot product of Equation (3) and $\dot{\boldsymbol{r}}$, we obtain the time derivative of $E(t)$ as follows:

$$\frac{d}{dt}E(t) = -\left(\frac{G\mu M_{\mathrm{S}}}{2|\boldsymbol{r}_{\mathrm{G}}|^3}\right)\frac{d}{dt}(|\boldsymbol{r}|^2) + 3G\mu M_{\mathrm{S}}\frac{(\boldsymbol{r}\cdot\boldsymbol{r}_{\mathrm{G}})(\dot{\boldsymbol{r}}\cdot\boldsymbol{r}_{\mathrm{G}})}{|\boldsymbol{r}_{\mathrm{G}}|^5}, \tag{24}$$

where

$$E(t) = \frac{\mu}{2}|\dot{\boldsymbol{r}}|^2 - \frac{G\mu M_{\mathrm{G}}}{|\boldsymbol{r}|}. \tag{25}$$

The first and second terms on the right side of Equation (24) are from the first- and second-order approximation terms in the Maclaurin expansion of Equation (2), respectively. Setting E_0 as the Moon's mechanical energy in the Earth-Moon system, we can write $E(t)$ as follows:

$$E(t) = E_0 + \Delta E_1(t) + \Delta E_2(t), \tag{26}$$

where $\Delta E_1(t)$ and $\Delta E_2(t)$ are the perturbative corrections of the Moon's mechanical energy, which correspond to the first and second terms on the right side of Equation (24), respectively. [3].

2.5.1 Correction of the Mechanical Energy of the Moon by the First Approximation

According to Equation (24), the time derivative of $\Delta E_1(t)$ is

$$\frac{d}{dt}\Delta E_1(t) = -\left(\frac{G\mu M_{\mathrm{S}}}{2|\boldsymbol{r}_{\mathrm{G}}|^3}\right)\frac{d}{dt}|\boldsymbol{r}|^2. \tag{27}$$

Expanding around $\boldsymbol{r}_0$ and $\boldsymbol{r}_{\mathrm{G0}}$ for $\boldsymbol{r}$ and $\boldsymbol{r}_{\mathrm{G}}$ in Equation (27), we extract the leading terms to derive

$$\frac{d}{dt}\Delta E_1(t) = -\left(\frac{G\mu M_{\mathrm{S}}}{2|\boldsymbol{r}_{\mathrm{G0}}|^3}\right)\frac{d}{dt}|\boldsymbol{r}_0|^2. \tag{28}$$

We substitute Equation (17) into Equation (28) and consider terms up to the first order of ϵ_0 and ϵ_{E0}, as $\epsilon_0 \sim \epsilon_{\mathrm{E0}} \ll 1$ [1]. We further obtain

$$\frac{d}{dt}\Delta E_1(t) = -2G\mu M_{\mathrm{S}}\frac{a_0^2}{A_0^3}\epsilon_0\dot{\psi}_{\mathrm{M}}\sin\psi_{\mathrm{M}}(t). \tag{29}$$

[3] $\Delta E_1(t) \sim \Delta E_2(t)$ suggests that the analysis should be performed up to at least a second-order approximation when considering the perturbation of the Sun in the Earth-Moon system.

By integrating the above equation, we obtain

$$\Delta E_1(t) = 2G\mu M_{\rm S}\frac{a_0^2}{A_0^3}\epsilon_0\cos\psi_{\rm M}(t). \tag{30}$$

The integral constant in Equation (30) can be set to zero without loss of generality as we investigate the periodicity of the Moon's orbit owing to the perturbation of the Sun.

2.5.2 Correction of the Mechanical Energy of the Moon by the Second Approximation

According to Equation (24), the time derivative of $\Delta E_2(t)$ is

$$\frac{d}{dt}\Delta E_2(t) = 3G\mu M_{\rm S}\frac{(\boldsymbol{r}_0\cdot\boldsymbol{r}_{\rm G0})(\dot{\boldsymbol{r}}_0\cdot\boldsymbol{r}_{\rm G0})}{|\boldsymbol{r}_{\rm G0}|^5}. \tag{31}$$

From Equations (14), (15), and (17), we derive

$$\begin{aligned}\frac{2}{3G\mu M_{\rm S}\dot{\psi}_{\rm M}}\frac{A_0^3}{a_0^2}\frac{d}{dt}\Delta E_2 = &-\sin 2\varphi \\ &+\frac{1}{2}\epsilon_0\Big(2\sin(2\varphi+\psi_{\rm M})+2\sin(2\varphi-\psi_{\rm M})+2\sin\psi_{\rm M}+\sin(\psi_{\rm M}+2\varphi)+\sin(\psi_{\rm M}-2\varphi)\Big) \\ &-\frac{3}{2}\epsilon_{\rm E0}\Big(\sin(2\varphi+\psi_{\rm E})+\sin(2\varphi-\psi_{\rm E})\Big),\end{aligned} \tag{32}$$

where we consider terms up to the first order of ϵ_0 and $\epsilon_{\rm E0}$. If we integrate the above equation, ΔE_2 becomes

$$\Delta E_2(t) = -\frac{3}{4}G\mu M_{\rm S}\frac{a_0^2}{A_0^3}C_E(t), \tag{33}$$

where

$$\begin{aligned}C_E(t) \equiv \cos 2\varphi + \epsilon_0\Bigg(&\frac{2\dot{\psi}_{\rm M}\cos(2\varphi+\psi_{\rm M})}{2\dot{\varphi}+\dot{\psi}_{\rm M}}+\frac{2\dot{\psi}_{\rm M}\cos(2\varphi-\psi_{\rm M})}{2\dot{\varphi}-\dot{\psi}_{\rm M}}+2\cos\psi_{\rm M} \\ &+\frac{\cos(\psi_{\rm M}+2\varphi)}{\dot{\psi}_{\rm M}+2\dot{\varphi}}+\frac{\cos(\psi_{\rm M}-2\varphi)}{\dot{\psi}_{\rm M}-2\dot{\varphi}}\Bigg)-3\epsilon_{\rm E0}\left(\frac{\dot{\psi}_{\rm M}\cos(2\varphi+\psi_{\rm E})}{2\dot{\varphi}+\dot{\psi}_{\rm E}}+\frac{\dot{\psi}_{\rm M}\cos(2\varphi-\psi_{\rm E})}{2\dot{\varphi}-\dot{\psi}_{\rm E}}\right).\end{aligned} \tag{34}$$

The integral constant in Equation (33) can be set to zero without loss of generality because we investigate the periodicity of the Moon's orbit owing to the perturbation of the Sun.

2.5.3 Perturbative Correction for the Moon's Mechanical Energy

According to Equations (26), (30), and (33), perturbative correction for the Moon's mechanical energy, $\Delta E(t)$ is

$$\Delta E(t) = G\mu M_{\rm S}\frac{a_0^2}{A_0^3}\left(2\epsilon_0\cos\psi_{\rm M}(t)+\frac{3}{4}C_E(t)\right). \tag{35}$$

We further derive $E(t)$ using E_0. In the Earth-Moon system, E_0 is represented as follows:

$$E_0 = -\frac{G\mu M_{\rm G}}{2a_0}, \tag{36}$$

where $E_0 < 0$ (see Section A.3). Moreover, we substitute Equation (17) into Equation (28) and consider the terms up to the first order of ϵ_0 and $\epsilon_{\rm E0}$, as $\epsilon_0 \sim \epsilon_{\rm E0} \ll 1$ [1]. Using Equations (26), (35), and (36), we obtain

$$E(t) = E_0\left[1+\lambda\left(2\epsilon_0\cos\psi_{\rm M}(t)+\frac{3}{4}C_E(t)\right)\right], \tag{37}$$

where

$$\lambda \equiv \left(\frac{a_0}{A_0}\right)^3\frac{M_{\rm S}}{M_{\rm G}}. \tag{38}$$

3 Results

In this section, we derive the Moon's orbit $r(t)$ owing to the Sun's perturbation in the Earth-Moon system. This study considers that the Moon's orbital semi-major axis $a(t)$ and eccentricity $\epsilon(t)$ are time-dependent in the system and are represented by the Moon's angular momentum $l(t)$ and mechanical energy $E(t)$.

Substituting Equations (22) and (37) into Equations (6) and (7), we obtain $a(t)$ and $\epsilon(t)$ as follows:

$$a(t) = a_0 \left(1 - \frac{3}{2} \frac{\dot{\psi}_{\mathrm{M}}}{\dot{\varphi}} \lambda \cos 2\varphi(t) \right), \tag{39}$$

$$\epsilon(t) = \epsilon_0 \left(1 - \frac{3}{4} \frac{\nu - \dot{\psi}_{\mathrm{M}} \lambda}{\dot{\varphi}} \cos 2\varphi(t) \right), \tag{40}$$

where

$$\epsilon_0 \sim 5 \times 10^{-2}, \quad \nu C_l(t) \sim 10^{-3}, \quad \lambda \sim 5 \times 10^{-3}, \tag{41}$$

which are considered up to the first-order approximations for ϵ_0, $\nu C_l(t)$, and λ [1]. Substituting Equations (39) and (40) into Equation (5), $r(t)$ becomes

$$r(t) = a_0 \left(1 - \epsilon_0 \cos \psi_{\mathrm{M}} - \frac{3}{2} \frac{\dot{\psi}_{\mathrm{M}}}{\dot{\varphi}} \lambda \cos 2\varphi(t) \right). \tag{42}$$

With $\varphi(t) = \psi_{\mathrm{M}}(t) - \psi_{\mathrm{E}}(t)$, the above equation can be rewritten as follows:

$$r(t) = a_0 \Big(1 - \epsilon'(t) \cos \psi_{\mathrm{M}}(t) \Big), \tag{43}$$

where

$$\epsilon'(t) \equiv \epsilon_0 \Big[1 + \frac{3}{2} \frac{\dot{\psi}_{\mathrm{M}}}{\dot{\varphi}} \frac{\lambda}{\epsilon_0} \Big(2 \cos \psi_{\mathrm{M}}(t) - \frac{1}{\cos \psi_{\mathrm{M}}(t)} \Big) \left(\cos 2\psi_{\mathrm{E}}(t) + \tan 2\psi_{\mathrm{M}}(t) \sin 2\psi_{\mathrm{E}}(t) \right) \Big], \tag{44}$$

and $\epsilon'(t)$ denote the time-dependent amplitude of $r(t)$; From Equation (43), we represent the time variation of the geocentric distance, as shown in Fig.1. The horizontal axis represents the days elapsed from January 1, 2021, and the vertical axis represents $r(t)/a_0$.

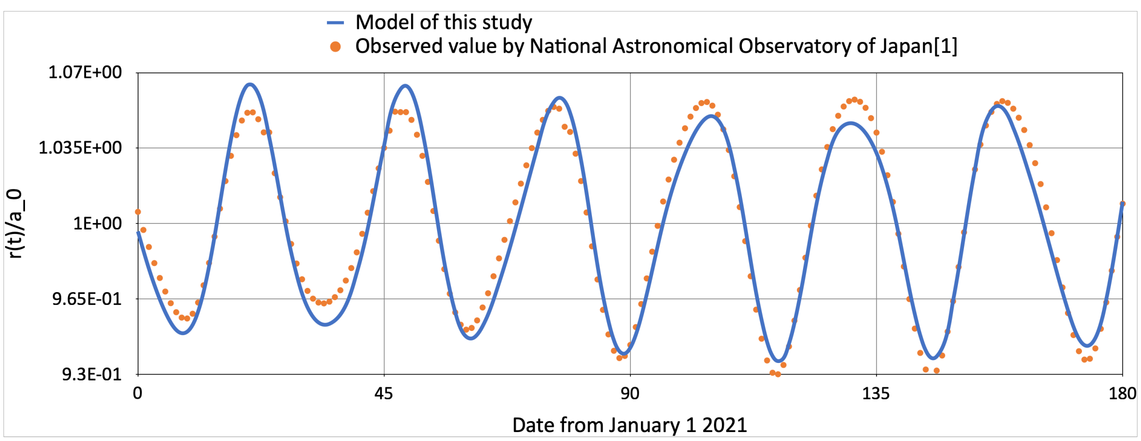

Figure 1: Time Variation of Geocentric Distance to the Moon's Orbital Semi-Major Axis Considering Solar Perturbations

4 Discussion

We consider time $t = t_n$ when the Moon is at perigee, implying that $\cos\psi_{\mathrm{M}}(t_n) = 1$ in Equations (43, 44). Then, $\epsilon'(t_n)$ becomes

$$\epsilon'(t_n) = \epsilon_0 \left(1 + \frac{3}{2}\frac{\dot{\psi}_{\mathrm{M}}}{\dot{\varphi}}\frac{\lambda}{\epsilon_0}\cos 2\psi_{\mathrm{E}}(t_n)\right), \tag{45}$$

where

$$\frac{3}{2}\frac{\dot{\psi}_{\mathrm{M}}}{\dot{\varphi}}\frac{\lambda}{\epsilon_0} \sim 0.16. \tag{46}$$

Because $\cos 2\psi_{\mathrm{E}}(t_n)$ in Equation (45) oscillates with a half-period of the anomalistic year, the local minima of $r(t)$ and $r(t_n)$ change within the half-year cycles. From Equation (46), we can infer that the local minima of the geocentric distance in the Earth-Moon system changes by up to $\pm 16\%$, which is consistent with the perturbation method in this study.

Additionally, as the Moon's orbital eccentricity at the perigee reaches its local maxima, $\cos\psi_{\mathrm{M}}(t)$ in Equation (43) and $\cos 2\psi_{\mathrm{E}}(t)$ in Equation (45) simultaneously become the local maxima. In this scenario, the Sun, Earth, and Moon are aligned in a straight line. However, the Moon's orbital eccentricity is at its minimum at the perigee when the Moon is perpendicular to the Sun with respect to the Earth.

These orbital changes are attributable to the tidal force of the Sun. As shown in Fig.(2), when the tidal force stretches the Earth-Moon system, the Moon's orbital eccentricity increases, whereas $r(t_n)$ decreases. Otherwise, the Moon's orbital eccentricity decreases, and $r(t_n)$ increases. This period is similar to the half-day tidal period of the Earth due to the tidal force of the Moon because the Moon's orbit is considered to change semiannually owing to the tidal force of the Sun. Therefore, this study explains the time evolution of the perigee by considering the Sun's perturbation in the Earth-Moon system.

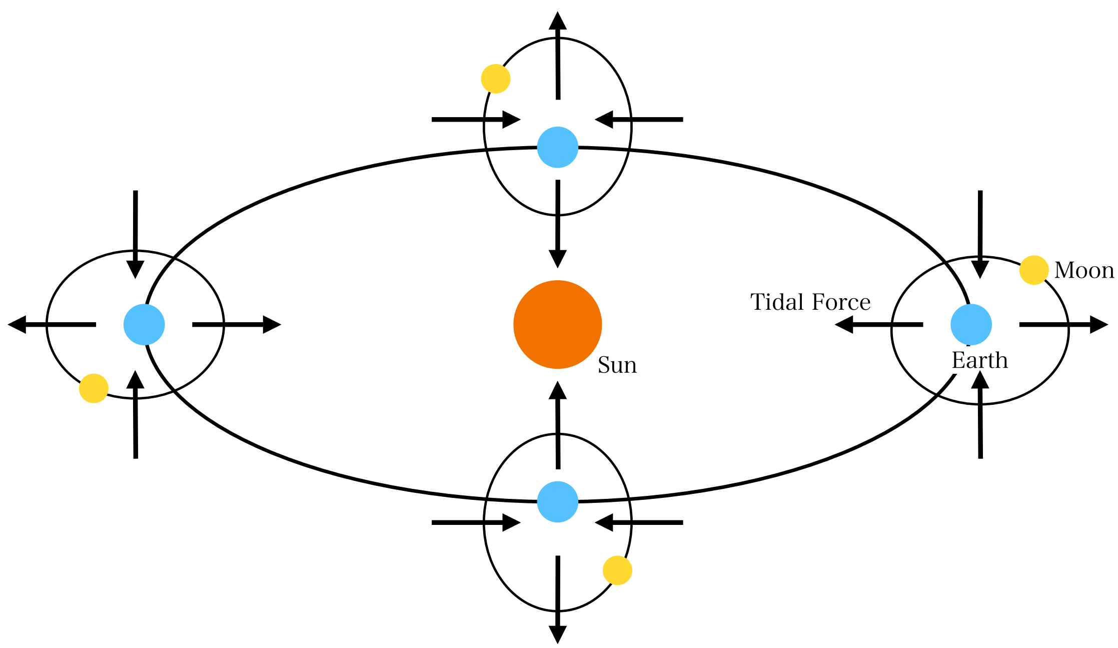

Figure 2: Sun's Tidal Force in the Earth-Moon System

5 Conclusion

We incorporate the Sun's perturbation into our analysis of the Earth-Moon system and the Moon's orbit. Our findings indicate that the Moon's orbital eccentricity undergoes changes in a six-month cycle owing to the Sun's tidal force. These results contribute to explaining familiar natural phenomena, such as the change in the visual diameter of supermoons. The results also show that tidal forces, which are closely related to our daily lives, significantly contribute to changes in the Moon's orbit. Therefore, this study offers valuable insights into the pedagogy of perturbation theory.

However, the Sun's perturbation, as modeled in this study, does not fully explain the Moon's orbit. This is evident as the amplitude of the geocentric distance observed in the time-dependent apogee, as shown in Fig.1, is less than what our model predicts. We attribute this discrepancy to the limitations of modeling the Earth, Moon, and Sun as point masses. In future research, we hope to overcome these limitations of the point mass approximation by constructing a more accurate model, which is expected to further enhance our understanding of the Sun's perturbation in the Earth-Moon system.

From an educational perspective in physics, this research provides practical insights and promotes the development of physical intuitions among students, providing an example of the interpretation of natural phenomena using perturbation theory. Additionally, refining our model to resemble the actual orbit of the Moon more closely makes it more applicable to physics education. These findings underscore a new perspective in physics education, emphasizing the significance of qualitatively understanding celestial body motion through perturbation theory.

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Data Access Statement

All relevant data are within the paper and its Supporting Information files.

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A Solutions to the two-body problem

In this appendix, we discuss the Moon's motion in the Earth-Moon system without considering the Sun's perturbation. We eliminate the perturbative terms from Equation (3) to obtain

$$\mu \ddot{\boldsymbol{r}}_0 = -G\mu M_{\mathrm{G}} \left(\frac{\boldsymbol{r_0}}{r_0^3} \right), \tag{47}$$

where $\boldsymbol{r}_0$ represents the position of the Moon with respect to the Earth in the Earth-Moon System.

A.1 The Moon's Angular Momentum

We demonstrate that the Moon's angular momentum l_0 in the Earth-Moon system is conserved. We take the cross-product of Equations (47) and $\boldsymbol{r_0}$ to derive

$$\mu \left(\boldsymbol{r}_0 \times \ddot{\boldsymbol{r}}_0 \right) = -G\mu M_{\mathrm{G}} \left(\frac{\boldsymbol{r}_0 \times \boldsymbol{r}_0}{r^3} \right) = \mathbf{0}. \tag{48}$$

Therefore, the time derivative of the Moon's angular momentum $\boldsymbol{l}_0 = \boldsymbol{r}_0 \times \dot{\boldsymbol{r}}_0$ becomes

$$\frac{d}{dt}\boldsymbol{l}_0 = \mu\frac{d}{dt}\left(\boldsymbol{r}_0 \times \dot{\boldsymbol{r}}_0\right) = \boldsymbol{0}. \tag{49}$$

This suggests that the Moon's angular momentum $\boldsymbol{l}_0$ is conserved; therefore, the motion of the Moon is confined to a single plane. Consequently, we set the 3-dimensional Cartesian coordinate system with the Moon's orbital plane as the xy plane and the direction of $\boldsymbol{l}_0$ as the z-axis positive direction, with Earth as the origin. Then, $\boldsymbol{r}_0\,(t)$ and $\boldsymbol{l}_0$ can be expressed as follows:

$$\boldsymbol{r}_0\,(t) = \begin{pmatrix} x_0(t) \\ y_0(t) \\ 0 \end{pmatrix}, \tag{50}$$

$$\boldsymbol{l}_0 = \begin{pmatrix} 0 \\ 0 \\ l_0 \end{pmatrix}. \tag{51}$$

In polar coordinates,

$$x_0(t) = r_0(t)\cos\psi(t), \tag{52}$$

$$y_0(t) = r_0(t)\sin\psi(t). \tag{53}$$

Therefore, the magnitude of $\boldsymbol{l_0}$ is expressed as follows:

$$\begin{aligned} l_0 &= x_0(t)\dot{y_0}(t) - y_0(t)\dot{x_0}(t) \\ &= r_0^2\dot{\psi}(t), \end{aligned} \tag{54}$$

where $\psi(t)$ is the anomaly of the Moon.

A.2 The Moon's Mechanical Energy

Next, we show that the Moon's mechanical energy E_0 in the Earth-Moon system is conserved. We take the dot product of Equation (47) and $\dot{\boldsymbol{r}}_0$ to derive

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{\boldsymbol{r}}_0^2 - \frac{G\mu M_{\mathrm{G}}}{r_0}\right) = 0 \Leftrightarrow \frac{d}{dt}E_0 = 0. \tag{55}$$

Therefore, the Moon's mechanical energy is conserved. We represent E_0 in the polar coordinate system and substitute $\dot{\psi}(t)$ into it using Equation (54).

$$\frac{1}{2}\mu\left\{\left(\frac{dr_0}{dt}\right)^2 + \frac{l_0^2}{r_0^2}\right\} - \frac{G\mu M_{\mathrm{G}}}{r_0} = E_0. \tag{56}$$

A.3 Orbital Semi-Major Axis and Eccentricity

According to Equation (54),

$$\frac{dr_0}{dt} = \frac{dr_0}{d\psi}\frac{d\psi}{dt} = \frac{l_0}{r_0^2}\frac{dr_0}{d\psi}. \tag{57}$$

We substitute the above into (56) to derive

$$\begin{aligned} \left(\frac{dr_0}{d\psi}\right)^2 =& \frac{2E_0}{\mu l^2}\left(r_0^2 + \frac{G\mu \mathrm{M_G}}{E_0}r_0 - \frac{\mu l_0^2}{2E_0}\right)r_0^2 \\ =& \frac{2E_0}{\mu l^2}(r_0 - r_1)(r_0 - r_2)r_0^2, \end{aligned} \tag{58}$$

where r_1 and r_2 $(r_1 < r_2)$ satisfy

$$r_1 + r_2 = -\frac{G\mu M_\mathrm{G}}{E_0}, \tag{59}$$

$$r_1 r_2 = -\frac{\mu l_0^2}{2E_0}. \tag{60}$$

Equation (58) indicates that by setting E_0 to negative, we can specify range r_0 to $r_1 \le r_0 \le r_2$, indicating that the orbit is a closed path. Considering $E_0 = -|E_0|$, we can rewrite Equations (59 and 60) as

$$r_1 + r_2 = \frac{G\mu M_\mathrm{G}}{|E_0|}, \tag{61}$$

$$r_1 r_2 = \frac{\mu l_0^2}{2|E_0|}. \tag{62}$$

Setting ϵ_0 and a_0 as the eccentricity and semi-major axis of the Moon's elliptical orbit, respectively, we can obtain the Moon's orbit as follows:

$$r_0(\psi) = \frac{a_0(1-\epsilon_0^2)}{1+\epsilon_0 \cos\psi}. \tag{63}$$

We perform a first-order approximation of the equation above using ϵ_0 as a small quantity to derive

$$r_0 = a_0(1 - \epsilon_0 \cos\psi), \tag{64}$$

When $\psi = 0, \pi$

$$r_0(0) = a_0(1-\epsilon_0) = r_1, \tag{65}$$

$$r_0(\pi) = a_0(1+\epsilon_0) = r_2. \tag{66}$$

According to Equations (61 and 62), the following hold true:

$$a_0 = \frac{GM_\mathrm{S} M_\mathrm{G}}{2|E_0|}, \tag{67}$$

$$\epsilon_0 = \sqrt{1 - \frac{2\mu l_0^2 |E_0|}{G^2 M_\mathrm{S}^2 M_\mathrm{G}^2}}. \tag{68}$$

Ultimately, we obtain a_0 and ϵ_0 as functions of l_0 and E_0. The distance between the Moon and the Earth can be determined by substituting Equation (68) into Equation (63).