A Problem with Cowling's Theorem

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Abstract Cowling's theorem is indispensable for the study of the magnetic fields of celestial bodies and conductive fluids. Cowling's paper argues that it is impossible that an axially symmetric field shall be self-maintained. However, the discussion of this paper uses Ohm's law equation, which does not include the electromotive force due to vector potential. It is unnatural to prove with a formula that does not include this physically existing electromotive force. Furthermore, since the fluctuation (growth) of the magnetic field is derived from a natural formula that includes the electromotive force, it is natural to think that such a phenomenon exists. Furthermore, the energetic considerations of the induction term of the equation allow the possibility of fluctuation (growth) of the magnetic field to be described in detail. Then, it can be found that the conversion of convective energy to electrical energy and from electrical energy to convective energy occurs simultaneously. Since the absolute value of these energies is the same, they cancel out and become zero, so at first glance nothing seems to happen. However, individual energies exist, and fluctuations (growth) are possible while the absolute value remains the same. Therefore, an increase in the axisymmetric magnetic field due to axisymmetric convection is possible.

Keywords: Cowling's theorem, dynamo theory, anti-dynamo theory, electromagnetic induction, conductive fluid, magnetohydrodynamics

1. Introduction

Hereinafter, unless otherwise explained, the explanation is limited to axisymmetric phenomena of conductive fluid. In addition, paper [1] of Cowling et al. is referred to as Cowling's paper.

Cowling's paper argues that it is impossible that an axially symmetric field shall be self-maintained (hereinafter also referred to as Cowling's theorem.) This is questionable.

Cowling's theorem is indispensable for the study of the magnetic fields of celestial bodies and conductive fluids. However, if you read the paper, you will find a problem. A passage from Cowling's paper has the following meaning: The first-order partial derivative of the stream function of the magnetic field is zero, and its second-order partial derivative is not zero. Applying these results to the magnetic maximum or minimum point (hereafter abbreviated as pole) causes a contradiction in the electromagnetic induction equation.

You will find strange points in this description. Upon examination, it turns out that Cowling's paper is unnatural. This explanation uses Ohm's law equation [4], which does not include the electromotive force [5] due to the vector potential. The equation that does not include an electromotive force due to a vector potential is unnatural. This method should not be able to verify correctly. Therefore, it seems wrong to use an equation that lacks the electromotive force. Furthermore, the possibility of increasing the magnetic field can be explained in detail by an equation that includes the electromotive force due to the vector potential.

To do this, we give energetic considerations to the induction term. Then, it can be found that the conversion of convective energy to electrical energy and from electrical energy to convective energy occurs simultaneously. Since the absolute value of these energies is the same, they cancel out and become zero, so at first glance nothing seems to happen. However, individual energies exist, and fluctuations (growth) are possible while the absolute value remains the same. That is, an increase in the magnetic field is possible.

Furthermore, Cowling's paper makes a similar statement from the comparison of the following two currents. One is the current calculated from the magnetic field around the cross-section of the current path, and the other is the current calculated from the magnetic field and convection in the cross-sectional region. Again, the consideration does not include the electromotive force due to the vector potential, so you will find the same as above.

2. Description of the problem

Some of the important parts of Cowling's paper are as follows: In Cowling's paper [2] in verses 2-4, the following meaning is stated: The first-order partial derivative of the stream function of the magnetic field is zero, and its second-order partial derivative is not zero. Applying these results to the magnetic maximum or minimum point causes a contradiction in the electromagnetic induction equation.

(Unless otherwise explained, symbols or similar symbols with the same meaning as in Cowling's paper are used in this chapter only.)

Here is the problem. In Cowling's theorem, the equation does not include an electromotive force due to the vector potential $\boldsymbol{A}$ and is calculated through Ohm's law as follows.

$$J = \sigma(c_\wedge \boldsymbol{H} - \text{grad } V)$$

It is added as shown below.

$$J = \sigma\left(c_\wedge \boldsymbol{H} - \frac{\partial \boldsymbol{A}}{\partial t} - \text{grad } V\right)$$

Using terms already derived in Cowling's paper, reflecting this in the electromagnetic induction equation and organizing it yields the following:

$$\frac{\partial A}{\partial t} = \frac{1}{4\pi\varpi\sigma}\left(\frac{\partial^2\psi}{\partial\varpi^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right) + \frac{1}{\rho\varpi^2}\left(\frac{\partial\phi}{\partial\varpi}\frac{\partial\psi}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial\varpi}\right) \tag{1}$$

What happens if we substitute $\frac{\partial\psi}{\partial z} = 0$, $\frac{\partial\psi}{\partial\varpi} = 0$ and $\frac{\partial^2\psi}{\partial^2\varpi^2} + \frac{\partial^2\psi}{\partial z^2} \neq 0$ in this equation? Of course, the second term on the right side is zero, but the first term is not zero. Therefore, the right side does not become zero, and the left side fluctuates. That is, even if there is a specific pole and its second-order partial derivative value is not zero, this equation is satisfied. It can be understood that the magnetic flux decreases due to attenuation according to the first term on the right side.

Is fluctuation limited to attenuation? Fluctuations should also be expected to increase. If ψ increases, there must be power generation somewhere in this equation. Since there is no permanent magnet, it is a self-excitation generator. If there is such a function, it would be the induction term (second term on the right side). However, the second term on the right side of Eq. (1) is zero, as described above.

However, the reason can be seen by energetic considerations of this term. Multiplying the parentheses in the second term by ψ:

$$\psi\left(\frac{\partial\phi}{\partial\varpi}\frac{\partial\psi}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial\varpi}\right) = \frac{\partial\phi}{\partial\varpi}\psi\frac{\partial\psi}{\partial z} - \frac{\partial\phi}{\partial z}\psi\frac{\partial\psi}{\partial\varpi}$$

Then, when transformed using partial integrals,

$$= -\phi\frac{\partial}{\partial\varpi}\left(\psi\frac{\partial\psi}{\partial z}\right) + \phi\frac{\partial}{\partial z}\left(\psi\frac{\partial\psi}{\partial\varpi}\right)$$

Furthermore, if we decompose this equation,

$$= -\phi\frac{\partial\psi}{\partial\varpi}\frac{\partial\psi}{\partial z} - \phi\psi\frac{\partial^2\psi}{\partial\varpi\partial z} + \phi\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial\varpi} + \phi\psi\frac{\partial^2\psi}{\partial z\partial\varpi} \tag{2}$$

become.

The first and third terms on the right side of this equation are both zero due to the conditions of the poles. The fourth and second terms on the right side correspond to the energy generated by the generator and the energy consumed by the motor. Since the second and fourth terms have the same absolute value, they are offset to zero. Here, power generation means that convective energy is converted into electrical energy, and consumption means that electrical energy is converted into convective energy. Note that consumption is different from conversion to Joule heat by attenuation terms. Therefore, since these two energies cancel each other, at first glance nothing seems to happen, but the magnetic field fluctuates (growths).

However, the above requires the assurance that the second-order partial derivative of these terms is not zero. Since the poles arise along the convection current, there is no single point in isolation. Like mountain ranges, there is a spread in the direction of convection flow and its width. In terms of contour lines, it is shaped like an elongated ellipse. It is rare for convection to be completely in the direction of the ϖ-axis or z-axis. In most cases, the longitudinal direction of the ellipse is inclined from the ϖ-axis and z-axis directions. At poles such as mountain ranges, the second-order partial derivative is not zero.

Cowling's paper does not mention the possibility of such magnetic field growth. I think Cowling's paper misses this.

In addition, Cowling's paper [3] explains the same thing by the following equation in verse 5, but it is incorrect:

$$H_0 s < \sigma C_0 H_0 S$$

Cowling's paper makes a similar statement from the comparison of the following two currents. The left-hand side is the current calculated from the magnetic field around the cross-section of the current path, and the right-hand side is the current calculated from the magnetic field and convection in the cross-sectional region. If it is a pole, this equation should hold, but since S has an infinitesimal order higher than s, the right-hand side approaches zero faster, so the inequality relationship is reversed. Therefore, it states that it contradicts being a pole. However, even this equation does not consider the electromotive force due to the vector potential.

If you consider its electromotive force,

$$H_0 s < \sigma(C_0 H_0 S - \frac{1}{\varpi}\frac{\partial\psi}{\partial t})$$

Become.

The second term in brackets is the electromotive force ($-\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{1}{\varpi}\frac{\partial\psi}{\partial t}$.) ψ is proportional to the magnetic flux inside the circular current path centered on the Z axis corresponding to this point and is independent of *S*. ψ is determined not only by the current in S but also by the influence of currents in other current paths. Even if *S* is infinitesimal, this inequality equation does not particularly indicate that ψ and the current go to zero. If there is a pole in *S*, it is the same and does not contradict the existence of the pole. Again, Cowling's paper misses this again.

3. Discussion

As mentioned above, Cowling's paper is wrong because it misses a point. However, it is also questionable whether this theorem can be easily refuted. I would like to consider slightly more about the points that bothered me.

Why does Cowling's paper not include electromotive force due to vector potential? Cowling's theorem includes the word "maintain." This may mean a stable (nonfluctuating) magnetic field. It may have been decided not to include this electromotive force as a condition under which the magnetic field is maintained (does not fluctuate). However, then the approach is wrong. This is because this electromotive force physically exists, so it has an impossible condition.

On the other hand, if the fluctuation of the left side of Eq. (1) is regarded as "not maintained", it is Cowling’s theorem. However, it is unlikely that Cowling's theorem denies only stability while acknowledging the growth of the magnetic field. Therefore, Cowling's paper misses the fact that it can be explained by including the electromotive force due to the vector potential. At the very least, it can be said that fluctuations (growth) of axisymmetric magnetic fields due to axisymmetric convection can occur.

4. Conclusion

I believe that the left side of Eq. (1) can fluctuate (growth) because the existence of such self-excited power generation is expected. Moreover, the energetic considerations of the induction term indicate its existence.

I think that there are many things that have not yet been elucidated about the mechanism of self-excited power generation of conductive fluids and its dynamic mechanism. I believe that some of them have been excluded because of Cowling's theorem, so they have not yet been fully examined. There may also be studies that struggle to conform to Cowling's theorem. I hope that we will be able to reconsider such research and explore it further.

Conflicts of interest.

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