Stability of Reeb Ordering by Interleaving Distance

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Abstract

The Reeb graph is instrumental in extracting topological features from contour plots. In this context, the Reeb ordering method offers both a natural discretisation and an algorithmic approach to compute a Reeb tree. Our main theorem establishes stability within the interleaving distance among order-compatible topological spaces. Our contributions are fourfold: we construct the reflector functor for quotient structures in ordered spaces, introduce generalised trees in poset terms, define branch completeness for graph-like posets, and prove a strong normalisation theorem for posets. Furthermore, our interleaving distance metric makes our stability estimate much finer than the preceding study.

Keywords: Topological data analysis, Reeb graph, interleaving distance, partially ordered space

1 Introduction

The interdisciplinary field of topological data analysis has seen remarkable growth as a meeting point between mathematical science and data analysis. One of the key instruments in this nexus is the Reeb graph, a proven indispensable tool for understanding the intricate topological structures in scalar data and contour plots. By abstracting and simplifying complex shapes, Reeb graphs provide a robust framework for capturing essential features in various data types, thereby serving many applications ranging from shape analysis to machine learning.

Research by Biasotti et al. (2008) has highlighted the application of Reeb graphs in shape analysis and computer graphics. Another critical feature of Reeb graphs is their stability, particularly under the interleaving metric as shown by de Silva et al. (2016). Also, Brown et al. (2021) exhibit a symbiotic relationship with Mapper graphs in statistical settings. In a historical context, Reeb graphs originate from Morse theory and continue to find applications in diverse fields. For instance, Yokoyama and Yokoyama (2020) utilised Reeb graphs for classifying two-dimensional Hamiltonian flows from the viewpoint of topological dynamical systems.

Our work focuses on the Reeb ordering method, a semi-discrete formulation introduced by Uda et al. (2019). This method was invented to realise Topological Flow Data Analysis and can handle both continuous and discrete data types. A regular grid is a standard input data structure in two-dimensional flow data analysis. This grid acts as a lattice graph, characterised by its repeated adjacency patterns between lattice points. Such grids are commonly generated through numerical simulations or observational methodologies. The advantage of the Reeb ordering method is the ability to deal with data types, including such grids. Intriguingly, the Reeb ordering can be equivalently constructed from a continuous real-valued function defined on a topological space. Moreover, this ordering is homeomorphic to its corresponding Reeb tree under specific conditions. Therefore, the Reeb ordering is a natural semi-discrete analogue of the Reeb graph.

In the realm of data analysis, stability is not just a theoretical luxury; it is a practical necessity, especially when data is noisy or incomplete. A stable method can provide reliable insights even when subjected to less-than-ideal data conditions. Accordingly, the cornerstone of this article is to investigate the stability properties of the Reeb ordering method, a semi-discrete approach to topological data analysis. Our study extends and refines existing research by focusing on the stability of Reeb ordering within the scope of order-compatible topological spaces. Unlike earlier works, we delve into the challenges brought about by the inherent semi-discreteness of the method.

Our contributions are not extensions of previous work; we introduce novel mathematical tools and techniques to navigate these challenges. Crucially, our main theorem establishes the stability of the Reeb ordering method within the context of interleaving distance and improves upon the stability inequalities identified in the earlier work by de Silva et al. (2016). In tackling these intricate issues, our research demanded inventive breakthroughs to provide resolutions.

For the reader's convenience, the article's structure is laid out as follows: Section 2 introduces the Reeb ordering method, followed by Section 3, which provides a detailed account of the theory of ordered spaces pertinent to our analysis. Section 4 extends conventional notions of trees to better suit the semi-discrete nature of the Reeb ordering. We introduce a novel mathematical tool in Section 5 that aids in dissecting the graph-like structures commonly encountered in ordered sets. Section 6 discusses the interleaving distance metric between partially ordered tree spaces and showcases related lemmas and propositions. Lastly, Section 7 shows the stability theorem in the framework of ordered tree spaces. For readers primarily interested in the main theorem and its proof, we direct their attention to Definition 46 of the smoothing functor and Theorem 62 of stability.

2 Reeb graphs and Reeb posets

Let X be a path-connected topological space and $f\colon X \to \mathbf{R}$ a continuous function. We define that, for any $x, y \in X$, an equivalence relation $x \sim y$ holds exactly when both x and y belong to the same path-connected component of a level set $f^{-1}(a)$ for some $a \in \mathbf{R}$. The Reeb graph of f is the quotient space $X/{\sim}$. Indeed, under certain assumptions (e.g. on the smoothness of f), we can regard $X/{\sim}$ as a 1-complex, namely a graph. If X is simply connected, $X/{\sim}$ is also simply connected; hence it is a tree. In such cases, $X/{\sim}$ is called a Reeb or contour tree.

We prepare some elementary terms and notions. Let X and Y be sets. A subset R of $X \times Y$ is called a binary relation over X and Y. Using infix notation, we write $x\ R\ y$ whenever $(x, y) \in R$ to represent an R-relation between elements $x \in X$ and $y \in Y$. A binary relation R is said to be homogeneous if $X = Y$. Let R be a homogeneous binary relation over X. We say R is a quasi-order, also known as a preoder, if R satisfies the laws of reflexivity and transitivity. A set equipped with a quasi-order is called a quasi-ordered set, also known as a preodered set or a proset for short. Let R^{op} denote the opposite relation of R; $R^{\mathrm{op}} := \{(y, x) \mid x\ R\ y\}$. If R is a quasi-order, the binary relation $R \cap R^{\mathrm{op}}$ over X satisfies the laws of reflexivity, transitivity, and symmetry; hence, an equivalence relation. We call $R \cap R^{\mathrm{op}}$ an equivalence relation associated with R. R is called a (partial) order if it is a quasi-order satisfying the law of antisymmetry. A set equipped with a partial order is called a partially ordered set or a poset for short. Assume R is symmetric; namely, $x\ R\ y$ whenever $y\ R\ x$. Note that we can regard (X, R) as an undirected graph where, for any $x, y \in X$, the relation $x\ R\ y$ is viewed as an undirected edge. We say X is R-connected if it is connected as a graph. For convention, any subset X' of X is assumed to be endowed with the restricted binary relation $R' := R \cap (X' \times X')$, and we regard (X', R') as a graph as well. Let $(X, \lesssim)$ be a quasi-ordered set. We introduce the downset closure operator $\downarrow$; for any $a \in X$, we define $\downarrow a := \{x \in X \mid x \lesssim a\}$. We similarly define the upset closure operator by $\uparrow a := \{x \in X \mid x \gtrsim a\}$.

Remark 1 (Notation for orders)**.** *We use both quasi-orders and partial orders at the same place frequently. In this article, we thus use the different infix notations to distinguish them for convention. We write $\lesssim$ for a quasi-order and $\leq$ for a partial order. This convention is not standard. However, it is helpful in our context because we often use the equivalence relation $\sim$ associated with a quasi-order $\lesssim$.*

We introduce a Reeb ordering now.

Definition 2 (Reeb (quasi-)order and Reeb poset)**.** *Let X be a path-connected space and $f\colon X \to \mathbf{R}$ continuous. For a path $p\colon [0, 1] \to X$, we say p is f-increasing if and only if $f \circ p$ is increasing; for all $0 \leq s \leq t \leq 1$, it holds $f \circ p(s) \leq f \circ p(t)$. We define, for $x, y \in X$, a quasi-order $x \lesssim_{f\uparrow} y$ holds if and only if there exists an f-increasing path from x to y in X. We call $\lesssim_{f\uparrow}$ a Reeb quasi-order of (X, f). Taking quotient by the equivalence relation $\sim_{f\uparrow}$ associated with $\lesssim_{f\uparrow}$, we obtain a poset $(\tilde{X}, \leq) := (X, \lesssim_{f\uparrow})/{\sim_{f\uparrow}}$, which we call a Reeb poset of (X, f). The quotient partial order $\leq_{f\uparrow}$ is called a Reeb (partial) order.*

Constant paths yield the reflexivity law $x \lesssim_{f\uparrow} x$, and concatenations of f-increasing paths do the transitivity law; namely, $x \lesssim_{f\uparrow} y$ and $y \lesssim_{f\uparrow} z$ imply $x \lesssim_{f\uparrow} z$. Furthermore, paths on a level set $f^{-1}(a)$ yield the equivalence relation $\sim_{f\uparrow}$ associated with $\lesssim_{f\uparrow}$, which coincides with the equivalence relation that defines the Reeb graph of f. Taking the quotient of a quasi-ordered set by its associated equivalence relation, we obtain a partially ordered set in general. We can think of this procedure as a poset reflection $\mathbf{QoSet} \to \mathbf{PoSet}$; namely, it is the left adjoint of the inclusion functor $\mathbf{PoSet} \hookrightarrow \mathbf{QoSet}$, where $\mathbf{QoSet}$ and $\mathbf{PoSet}$ denote the categories of quasi-ordered sets and partially ordered sets, respectively. Let $q\colon X \to \tilde{X}$ denote the quotient map from X to the Reeb poset $\tilde{X}$ of (X, f). Note that the quotient induces a continuous function $\tilde{f}\colon \tilde{X} \to \mathbf{R}$ from f; namely, $f = \tilde{f} \circ q$. It is straightforward to see that the operation taking the Reeb poset is idempotent. In other words, the Reeb quasi-order $\lesssim_{\tilde{f}\uparrow}$ of $(\tilde{X}, \tilde{f})$ equals the Reeb partial order $\leq$ of (X, f), and it must satisfy the anti-symmetry. Conversely, the Reeb quasi-order $\lesssim_{f\uparrow}$ can be viewed as the inverse image of $q \times q$ of the Reeb ordering $\leq$ of the Reeb graph. Indeed, true generally holds this relation between a quasi-order and a quotient order via the inverse image under the poset reflection.

Next, we explain the (semi-)discretisation or approximation of Reeb posets. We want to obtain Reeb graphs from discrete data somehow. Let us consider a connected graph (X, R) with real values assigned to vertices; $f\colon X \to \mathbf{R}$. By replacing a continuous path with a discrete one in the sense of graph theory, we may introduce the similar notions of Reeb posets regarding (X, R, f). However, the equivalence relation $\sim_{f\uparrow}$ is not helpful when dealing with such discrete structures. In most cases, elements in level sets are isolated from each other. Therefore, the quotient set $X/{\sim_{f\uparrow}}$ does not extract much topological information. One way is restricting data structures to be only triangular meshes so we can use a continuous linear interpolation of the scalar function f. Indeed, many preceding studies have taken this approach both theoretically and computationally. Another way is approximating the equivalence relation associated with the Reeb quasi-order in a certain sense. The advantage of this approach is that we can compute the Reeb poset regardless of data structures. The disadvantage is that the approximation works mathematically well only when the graph comes from a spatial discretisation of a simply connected space. We will revisit these aspects later. Though the basic idea and the related algorithm have already been explained in Uda et al. (2019), as it is written only in Japanese, we describe the latter approach in detail.

When we deal with binary relations like orders, it is often helpful and convenient for us to distinguish its infix symbolic notation and its graph as a subset. We thus introduce, for any binary relation $\bullet\ R\ \bullet$ over X and Y, the notation of its graph, $(R) \coloneqq \{(x, y) \in X \times Y \mid x\ R\ y\}$. For example, for a partial order $\bullet \leq \bullet$ on X, its graph is represented by $(\leq) \subset X^2$. Remind that, mathematically speaking, they are the same things, and only the notation matters just for readability.

Definition 3 (Discretised version of Reeb poset)**.** *Let (X, R) be a connected graph (X, R) and $f\colon X \to \mathbf{R}$. We define that, for any $x, y \in X$, a quasi-order $x \lesssim_{-} y$ holds exactly when both x and y belong to the same path-connected component of the sublevel set $f^{-1}{\downarrow} f(y)$. Similarly, we define a quasi-order $x \lesssim_{+} y$ holds exactly when both x and y belong to the same path-connected component of the superlevel set $f^{-1}{\uparrow} f(x)$. We* *set a quasi-order* $(\lesssim_{\pm}) \coloneqq (\lesssim_{-}) \cap (\lesssim_{+})$*, which we call a Reeb quasi-order of* (X, R, f)*. We then take the poset reflection of the quasi-ordered set* $(X, \lesssim_{\pm})$*; namely, we take the quotient order* $(\tilde{X}, \leq_{\pm}) \coloneqq (X, \lesssim_{\pm})/{\sim_{\pm}}$ *where* $(\sim_{\pm}) \coloneqq (\lesssim_{\pm}) \cap (\gtrsim_{\pm})$ *is the equivalence relation associated with* $\lesssim_{\pm}$*. We call* $\leq_{\pm}$ *a Reeb (partial) order of* (X, R, f)*. We also call* $(\tilde{X}, \leq_{\pm})$ *a Reeb (partially) ordered set or a Reeb poset for short.*

By the way, $(X, \lesssim_{+})/{\sim_{+}}$ can be viewed as a merge tree of superlevel sets and $(X, \lesssim_{-})/{\sim_{-}}$ as that of sublevel sets. Hence, we can compute these quasi-orders via 0-dimensional persistent homology of superlevel or sublevel filtrations. We can combine these two to compute the Reeb quasi-order by a combinatorial algorithm, as explained in Uda et al. (2019). When we want to emphasise discrete or algorithmic aspects and distinguish them from Reeb posets in continuous settings, we use the term "Reeb ordering (method)".

Whereas level sets determine Reeb graphs, sublevel and superlevel sets determine the Reeb ordering method. Schematically speaking, if we combine topological information of both sublevel sets and superlevel sets, we can obtain or approximate one of the level sets. Indeed, the above definition coincides with the Reeb tree of a simply connected space. Let us see how this idea is justified. Let X be a simply connected space, $f\colon X \to \mathbf{R}$ a continuous function. We assume that X admits of triangulation on which f is piecewise linear. This assumption was imposed only for simplicity, and it may be altered by some other conditions so that we can justify the below discussion using homology. We take a triangulation $\mathcal{T}$ with $|\mathcal{T}| \xrightarrow{\sim} X$. Let R denote the neighbourhood relation over X via the realisation of the closed stars in the triangulation $\mathcal{T}$. Obviously, X is R-connected as X is path-connected. We apply the Reeb ordering method to (X, R, f). Consider the Reeb quasi-order $\lesssim_{\pm}$ of (X, R, f). Take $x, y \in X$ and assume $x \sim_{\pm} y$. By definition, $f(x) = f(y)$ must hold. Define $a \coloneqq f(x)$, $A \coloneqq f^{-1}{\downarrow}a$, and $B \coloneqq f^{-1}{\uparrow}a$. We have the Mayer–Vietoris exact sequence,

$$H_1(X) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0. \tag{1}$$

Here, $H_{\bullet}$ denotes the homology functor over the coefficient ring $\mathbf{Z}$. Since X is simply connected, we have $H_0(X) = \mathbf{Z}$ and $H_1(X) = 0$. Hence, (1) is short exact, from which it follows

$$H_0(A) \oplus H_0(B) \cong \mathbf{Z} \oplus H_0(A \cap B).$$

Since $x \lesssim_{-} y$ (resp. $x \lesssim_{+} y$), there is some path-connected component of the sublevel set A (resp. the superlevel set B), to which both x and y belong. It follows that the induced elements $[x]$ and $[y]$ coincide in $H_0(A)$ (resp. in $H_0(B)$), and hence the corresponding elements in $H_0(A \cap B)$ do as well. Noting that $A \cap B = f^{-1}(a)$, both x and y belong to the same connected component of the level set. Hence, $\sim_{\pm}$ implies $\sim$. The converse is trivially true as well, by definition. In conclusion, the Reeb poset $X/{\sim_{\pm}}$ coincides with the Reeb tree $X/{\sim}$. In such a way, we may regard the Reeb ordering method as a semi-discretisation of the Reeb graph.

Here are some remarks regarding discretisation. First, the Reeb ordering method does not require triangulation through the computation algorithm. The definition of the Reeb quasi-order of (X, R, f) depends only on the adjacency R and the scalar values f. Still, as seen in the above discussion, we may apply the Reeb ordering method to data with a semi-discrete structure.

Second, we do not require our input graph structure to be geometrically consistent. For example, we may even use a non-planar graph to represent a scalar function on a plane. Such a graph might have self-intersections when realised on a plane, causing no problems with the Reeb ordering method. Lattice graphs with many self-intersection adjacencies are often helpful in the Reeb ordering method. Observe that if an input graph has more and more edges, the resulting merge trees become more straightforward, and hence, small topological noises tend to disappear.

Third, the Reeb ordering method also applies to a spatial discretisation of a multiply connected space. There are two workarounds to avoid the restriction of simply connectedness. One is restricting the equivalence relation more so that multiply connectedness is considered. The definition of $\sim_\pm$ relies upon the existence of an intermediate level set between sublevel and superlevel sets, which is justified only when the underlying space is simply connected. In general, finding an appropriate contour is impossible if a part of sublevel and superlevel sets surround some non-trivial generators or halls. Thus, considering information about such generators in the equivalence relation, we may take a more suitable quotient instead. However, the computational cost would be problematic. Another is dividing the input space into several parts. This approach has already been proposed for Reeb graphs by Doraiswamy and Natarajan (2013). We first divide an input manifold into a set of subvolumes with Reeb graphs having no cycles, which the efficient contour tree algorithm can compute. Then, we combine these trees and merge them at each cut area to reconstruct the Reeb graph of the whole manifold. Extending their approach to the Reeb poset is straightforward by replacing the contour tree method with the Reeb ordering method.

Observing such relationships between Reeb graphs and Reeb posets, it is worth paying attention to the combination of topology and partial order. In the next section, we describe the theory of partially ordered spaces to deal with the compatibility of topology and partial order.

3 Ordered spaces

We first define the following notions.

Definition 4 (Spaces endowed with binary relations)**.** *Let* $(X, \mathcal{O}_X)$ *be a topological space and* R_X *a homogeneous binary relation over* X*.*

- *We call* $(X, \mathcal{O}_X, R_X)$ *a relational space.*
- *We call* $(X, \mathcal{O}_X, R_X)$ *a quasi-ordered space if* R_X *is a quasi-order.*
- *We call* $(X, \mathcal{O}_X, R_X)$ *a (partially) ordered space if* R_X *is a partial order.*
- *We say* R_X *is closed if it is closed as a subset of* X^2 *in the product topology.*
- *We say an ordered space* $(X, \mathcal{O}_X, \leq)$ *has closed compatibility if* $(\leq)$ *is closed.*
- *An ordered space with closed compatibility is called a CC-pospace for short.*

Remark 5 (Terms of relational spaces, related concepts, and their ambiguity)**.** *The above terms are not commonly used in literature, but we have introduced them carefully to avoid the ambiguity of these concepts. Significantly, some remarks on the term pospace are needed. First, a POSET stands for a Partially Ordered SET and a PROSET for a PReordered SET. These terms are well-known. Thus, it is natural to think of a POSPACE standing for a Partially Ordered SPACE. However, the* *term pospace is used to refer to different concepts in literature; a pospace may refer to a quasi-ordered space, a partially ordered space, and, most often, a partially ordered space with closed compatibility. Also, some authors refer to a prospace as a preordered space. Because we will deal with both a quasi-ordered space and a CC-pospace, we do not use the term pospace and always use a CC-pospace instead. Furthermore, we prefer a quasi-order to a preorder to distinguish them just in the first letter.*

Remark 6 (Compatibility of topology and order)**.** *There are some known concepts regarding the compatibility of topology and order other than closed compatibility. Here are some examples: an order topology, an interval topology, a Dedekind closedness, an upper semi-closedness, a lower semi-closedness, semi-closedness, a specialisation order, an Alexandroff topology, a (directed-)complete partial order, and a Scott topology. An interval topology is an important example, especially in our application of interest. Let* X *be a poset. An interval topology* $\mathcal{O}_X$ *on* X *is generated by* $\{\downarrow x \mid x \in X\} \cup \{\uparrow x \mid x \in X\}$ *as a subbase for closed sets. (See Birkhoff (1940) for instance.) If* X *is totally ordered, the interval topology* $\mathcal{O}_X$ *coincides with the order topology, and* $(X, \mathcal{O}_X, \leq)$ *is a CC-pospace. Also, the standard topology on an Euclidean line* $\mathbf{R}$ *is an interval topology associated with the standard partial order* $\leq$*. Although we think of general CC-pospaces in theory, the readers may assume that an ordered set is endowed with an interval topology in most typical situations. Indeed, Wolk (1960) has shown a necessary and sufficient condition for a poset to possess a unique order-compatible Hausdorff topology. The important thing is that an interval topology is always order-compatible, owing to Wolk's definition. In addition, Wolk also has presented a moderate sufficient condition for a poset to be metrisable in its interval topology. Since our attention is mainly paid to posets with graph-like structures, they usually have their intrinsic metric structures. Thus, it is natural to suppose most CC-pospaces are equipped with their interval topologies in our context.*

Remark 7 (Properties of CC-pospaces)**.** *Let* $(X, \mathcal{O}_X, \leq)$ *be a CC-pospace. Equivalently, the subset* $(\nleq)$ *of* X^2 *is an open set, which is a trivial but important property; for any points* $x, y \in X$ *with* $x \nleq y$*, we can take open neighbourhoods* U *of* x *and* V *of* y *that separate* x *and* y *and that, for all* $u \in U$ *and* $v \in V$*, we have* $u \nleq v$*. Actually when we use this property to show something, taking neighbourhoods in this way is verbose; using the notation of graphs, we can state instead that, for any* $(x, y) \in (\nleq)$*, there exists an open neighbourhood* $U \times V \subset (\nleq)$ *of* (x, y)*. Note that* $(\nleq)$ *is included in* $(\neq)$*; hence such* U *and* V *are disjoint. Also, it follows that* X *is a Hausdorff space as the diagonal set* $(\leq) \cap (\geq)$ *is closed. A CC-pospace requires the compatibility of topology and partial order. Indeed,* X *is a CC-pospace if and only if any converging net in* X *preserves the order; namely, for any converging nets* $(x_\lambda)_\lambda$ *and* $(y_\lambda)_\lambda$ *in* X *such that* $x_\lambda \leq y_\lambda$ *for all* λ*, we have* $\lim_\lambda x_\lambda \leq \lim_\lambda y_\lambda$*.*

3.1 The category of CC-pospaces CCPoHaus

Let **CCPoHaus** denote the category of CC-pospaces; namely, its objects are CC-pospaces, and its morphisms are continuous order-preserving maps between CC-pospaces. We will see that **CCPoHaus** is cocomplete. For this end, we first construct a CC-pospace reflector. Using the reflector's universality, we can easily prove that the category **CCPoHaus** has coequalisers.

Let X be a topological space and $\lesssim$ a quasi-order on X; the binary relation $\lesssim$ is reflexive and transitive. Note that we do not assume any compatibility condition on quasi-ordered spaces, contrasted with CC-pospaces. Let **QoTop** denote the category of quasi-ordered spaces; its objects are quasi-ordered spaces, and its morphisms are continuous order-preserving maps. We then introduce the following notion, CC-pospace reflection, to see the relation between **QoTop** and **CCPoHaus**, similar to that of **Top** and **Haus**, the categories of topological spaces and Hausdorff spaces, respectively.

Definition 8 (Pospace reflection). *Let X be a quasi-ordered space, $\tilde{X}$ a CC-pospace and $q\colon X \to \tilde{X}$ a morphism in* **QoTop**. *We say that $(\tilde{X}, q)$ is a CC-pospace reflection of X if, for any CC-pospace Y and any morphism $f\colon X \to Y$ in* **QoTop**, *there exists a unique morphism $\tilde{f}\colon \tilde{X} \to Y$ in* **CCPoHaus** *that factors f through q; that is to say, $f = \tilde{f} \circ q$.*

By the universality, the CC-pospace reflection is unique up to isomorphisms if it exists. In fact, any quasi-ordered space has its CC-pospace reflection. Categorically speaking, the CC-pospace reflection is the left adjoint of the inclusion functor **CCPoHaus** $\hookrightarrow$ **QoTop**, from which the term "CC-pospace reflection" derives. The following proposition ensures the existence of CC-pospace reflection, whose construction is similar to that of Hausdorffification, or Hausdorff reflection, which is the left adjoint of the inclusion **Haus** $\hookrightarrow$ **Top**. Readers are referred to the bachelor's thesis by van Munster (2014) for the construction of the Hausdorff reflection. Other references can be found in van Munster (2017) for its property preserving homotopy and Osborne (2014) for the general construction of reflectors. Note that, in addition, we have to check the closed compatibility of an ordered space in the following proof.

Proposition 9 (Existence of CC-pospace reflector). *For any quasi-ordered space X, there exists such an equivalence relation $\sim$ that the quotient $X/{\sim}$ is the CC-pospace reflection of X. Furthermore, the correspondence $\mathcal{P}\colon X \mapsto X/{\sim}$ naturally forms a functor* **QoTop** $\to$ **CCPoHaus**.

Before we prove the proposition, we introduce the following notion of a quasi-order for convention, which is often valuable for our contexts.

Definition 10 (Climbing Order). *Let X be a set, Y a quasi-ordered set, $R \subseteq X^2$ a binary relation over X, and $f\colon X \to Y$ a map from X to Y. We define, for all $x, x' \in X$, a quasi-order $x \lesssim_{f\uparrow} x'$ holds if and only if there exists such a path $(x_i)_{i=0}^n$ in (X, R) from x to x' along which the values of f are increasing; namely,*

$$x = x_0 \mathrel{R} x_1 \mathrel{R} \dots \mathrel{R} x_n = x', \qquad f(x_0) \lesssim f(x_1) \lesssim \dots \lesssim f(x_n).$$

Equivalently, the quasi-order $\lesssim_{f\uparrow}$ is nothing other than the transitive closure $R^*_{f\uparrow}$ of $R_{f\uparrow}$, where $R_{f\uparrow} \coloneqq \{(x, y) \in R \mid f(x) \lesssim f(y)\}$. We call $\lesssim_{f\uparrow}$ the (f-)climbing order on (X, R). If a climbing order is a partial order, we use the notation $\leq_{f\uparrow}$ instead.

As is clear by definition, a climbing (quasi-)order is another analogue to a Reeb (quasi-)order. We used the same notation $\lesssim_{f\uparrow}$ for a Reeb quasi-order in Section 2. However, we hereafter use the notation $\lesssim_{f\uparrow}$ to refer to the climbing (quasi-)order only. Note that the choice of the codomain Y of f is general in Definition 10, contrasted with the Reeb (quasi-)order.

Remind that, for a quasi-ordered set X and an equivalence relation $\sim$ on X, the quotient quasi-order $\lesssim_q$ on the quotient set $X/{\sim}$ is the minimum quasi-order that makes the quotient map $q\colon X \twoheadrightarrow X/{\sim}$ order-preserving. Equivalently, for any $x, y \in X$, if $q(x) \lesssim_q q(y)$, then there exist such sequences $(x_i)_{i=0}^n$ and $(y_i)_{i=0}^n$ in X that

$$x = x_0 \lesssim y_0 \sim x_1 \lesssim y_1 \sim \cdots \lesssim y_{n-1} \sim x_n \lesssim y_n = y.$$

Applying q, we get the following increasing sequence in $X/{\sim}$,

$$q(x_0) \lesssim_q q(y_0) = q(x_1) \lesssim_q q(y_1) = \cdots \lesssim_q q(y_{n-1}) = q(x_n).$$

Hence, setting $R \coloneqq ({\lesssim}) \cup ({\sim})$, the q-climbing order $\lesssim_{q\uparrow}$ on (X, R) coincides with the inverse image of $q \times q$ of $\lesssim_q$. In other words, we have, for all $x, y \in X$,

$$x \lesssim_{q\uparrow} y \quad \text{if and only if} \quad q(x) \lesssim_q q(y).$$

Thus, for a function $f\colon X \to Y$ to a quasi-ordered set in general, the f-climbing order will be a valuable tool to describe a quasi-order on the domain X, which somehow inherits the properties of the quasi-order of the codomain Y via f.

Now, we prove the existence of the CC-pospace reflector.

Proof of Proposition 9. Let $(X, \mathcal{O}_X, \lesssim)$ be a quasi-ordered space. We introduce a binary relation $\sim$ over X as follows. Let $x, y \in X$. Define $x \sim y$ holds if and only if, for all $Y \in$ **CCPoHaus** and all morphisms $f\colon X \to Y$ in **QoTop**, we have $f(x) = f(y)$. Note that the axiom of specification ensures the existence of the subset $({\sim})$ of X^2. Obviously, $\sim$ is an equivalence relation. Take the quotient and set $(\tilde{X}, \mathcal{O}_{\tilde{X}}, \leq) \coloneqq (X, \mathcal{O}_X, \lesssim)/{\sim}$. Let $q\colon X \to \tilde{X}$ be the quotient map. By definition of quotient topology and quotient quasi-order, q is a continuous order-preserving map. We show $(\tilde{X}, q)$ is a CC-pospace reflection of X.

First, we briefly confirm that the quotient quasi-order $\leq$ is a partial order. Recall that $x \lesssim_{q\uparrow} y$ if and only if $q(x) \leq q(y)$. Immediately, it follows that $x \sim_{q\uparrow} y$ if and only if $q(x) = q(y)$, where $\sim_{q\uparrow}$ denotes the equivalence relation associated with $\lesssim_{q\uparrow}$. Thus, $\sim_{q\uparrow}$ is nothing other than $\sim$, and $(X, \lesssim)/{\sim}$ coincides with the quotient poset $(X, \lesssim_{q\uparrow})/{\sim_{q\uparrow}}$, concluding $\leq$ is a partial order.

Second, we check the universality of CC-pospace reflection, which induces $\tilde{f}$ from f. The universality itself is almost trivial by that of quotient topology. Especially, $\tilde{f} = f \circ q^{-1}\colon \tilde{X} \to Y$ is a well-defined continuous order-preserving map. Thus, the remaining thing we have to show is that $\tilde{X}$ is a CC-pospace. Take $x, y \in X$ with $q(x) \neq q(y)$ arbitrarily. By definition, there are some CC-pospace Y and morphism $f\colon X \to Y$ satisfying $f(x) \neq f(y)$. We may assume $f(x) \not\leq f(y)$ without loss of generality. As mentioned above, we can define a continuous order-preserving map $\tilde{f} = f \circ q^{-1}\colon \tilde{X} \to Y$. Since Y is a CC-pospace, $f(x)$ and $f(y)$ can be separated by some open neighbourhoods U of $f(x)$ and V of $f(y)$ with $U \times V \subseteq ({\not\leq})$. This implies $\tilde{f}^{-1}(U) \times \tilde{f}^{-1}(V) \subseteq ({\not\leq})$ as $\tilde{f}$ preserves the order. Hence, $q(x)$ and $q(y)$ are separated by the open neighbourhoods $\tilde{f}^{-1}(U)$ and $\tilde{f}^{-1}(V)$. In conclusion, the subset $({\not\leq})$ of $\tilde{X}^2$ is open; namely, $\tilde{X}$ is a CC-pospace.

Finally, we briefly check the functoriality of $\mathcal{P}$. For any $X \in \mathbf{QoTop}$, let $q_X : X \to \mathcal{P}X$ denote the quotient map associated with the reflection. Then, for any morphism $\varphi\colon X \to Y$ in **QoTop**, we define a morphism in **CCPoHaus** as

$$\mathcal{P}\varphi \coloneqq q_Y \circ \varphi \circ q_X^{-1} : \mathcal{P}X \to \mathcal{P}Y.$$

Obviously this correspondence preserves the compositions; namely, for all $\varphi\colon X \to Y$ and $\varphi'\colon Y \to Z$ in **QoTop**,

$$\mathcal{P}\varphi' \circ \mathcal{P}\varphi = q_Z \circ \varphi' \circ q_Y^{-1} \circ q_Y \circ \varphi \circ q_X^{-1} = \mathcal{P}(\varphi' \circ \varphi).$$

The identity law $\mathcal{P}\mathrm{id}_X = \mathrm{id}_{\mathcal{P}X}$ is trivial by definition. Hence, $\mathcal{P}$ is a functor. □

Proposition 11 (Cocompleteness of **CCPoHaus**). *The category* **CCPoHaus** *is cocomplete.*

Proof. The existence of infinite coproducts is trivial. We will check the existence of coequalisers in **CCPoHaus**. Take parallel morphisms $\varphi, \psi\colon X \to Y$ in **CCPoHaus** arbitrarily. Let $Z \coloneqq \mathcal{P}(Y/{\sim})$, where $\sim$ denotes the minimum equivalence relation satisfying $\varphi(x) \sim \psi(x)$ for all $x \in X$. Then, Z is the coequaliser of φ and ψ. Indeed, the universality of the coequaliser follows from those of the quotient set $Y/{\sim}$ and of the CC-pospace reflection. As **CCPoHaus** admits of infinite coproducts and finite coequalisers, any diagram in **CCPoHaus** has its colimit. □

Example 12 (Incompatible combination of topology and order). *We give some non-trivial examples of CC-pospace reflections.*

- *Let $q\colon \mathbf{R} \to \mathbf{R}/\{\pm 1\} =: X$. We regard $X \in$* **QoTop** *endowed with the quotient topology and order. We have $q(-1) \lesssim q(x) \lesssim q(1) = q(-1)$ for all $-1 \leq x \leq 1$; in other words, we can not distinguish any elements in the interval $[-1, 1]$ by the quasi-order on X. Hence, we get $\mathcal{P}X \cong (-\infty, -1) \cup \{0\} \cup (1, \infty)$.*
- *Let $X \coloneqq \mathbf{R}$ be endowed with the standard order of real numbers and the topology generated by the following base.*

$$\mathcal{B} = \{(a, b) \subset \mathbf{R} \mid \mathbf{Z} \cap (a, b) = \emptyset\} \cup \{(a, b) + \mathbf{Z} \subset \mathbf{R} \mid \mathbf{Z} \cap (a, b) \neq \emptyset\}.$$

  Then, we get $\mathcal{P}X = \{0\}$ because we can not separate any pair of different integers by open neighbourhoods. Note that although $X/\mathbf{Z}$ is a Hausdorff space, the quotient order on $X/\mathbf{Z}$ satisfies $x \lesssim y$ for all $x, y \in X/\mathbf{Z}$; hence, the order can not distinguish any elements.
- *Set*

$$X \coloneqq \mathrm{colim}_{\mathbf{PoSet}}\{(0, 2)_a \leftarrow (0, 1) \to (0, 2)_b\}, \tag{2}$$

  where the morphisms are inclusion maps, and the subscripts a, b are labels to distinguish the two open intervals. Consider the quotient topology on X induced by the quotient map $q\colon (0, 2)_a \cup (0, 2)_b \twoheadrightarrow X$, where the disjoint union is regarded as a coproduct of subspaces of $\mathbf{R}$ in **Top**. *Then, X is not Hausdorff, as we can not separate 1_a and 1_b by open neighbourhoods. Thus, X is not a CC-pospace either.*

Take a pospace reflection of X, or equivalently, take a colimit in **CCPoHaus** *instead and set*

$$X' \coloneqq \mathcal{P}X \cong \operatorname{colim}_{\mathbf{CCPoHaus}}\{(0,2)_a \leftarrow (0,1) \rightarrow (0,2)_b\}. \tag{3}$$

Then, 1_a and 1_b must be identified in X'. By the way, we can also make X a CC-pospace by generating the topology with open sets of the form $[1,t)_a$ and $[1,t)_b$ added to the quotient topology. This topology coincides with the interval topology generated by a set of all closed intervals in X as a subbase for closed sets. However, the quotient map q to X is no longer continuous in this topology.

As we see in these examples an incompatible combination of topology and order often destroys those structures too much after taking the CC-pospace reflection. From another viewpoint, however, the CC-pospace reflection ensures that the quotient map is structure-preserving. Hence, when we take the quotient of CC-pospaces, we need to check whether the structures are kept as desired carefully.

The following lemma ensures we can safely coequalise two totally ordered subsets in a CC-pospace without destroying the structures. We will impose a technical assumption to guide partial orders along a given height function on the CC-pospace. To this end, we consider a slice category whose objects are pairs of CC-pospaces and continuous real-valued functions. For a category $\mathbf{C}$ and $a \in \mathbf{C}$, let $\mathbf{C}_{/a}$ denote the slice category of $\mathbf{C}$ over a. See Appendix A for the definition of slice categories.

Lemma 13 (Coequaliser of compact chains)**.** *Let $(C,f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ be a totally ordered compact CC-pospace. Let $\varphi_0, \varphi_1 \colon (C,f) \to (X,g)$ be parallel morphisms in $\mathbf{CCPoHaus}_{/\mathbf{R}}$. Let R denote the minimum equivalence relation on X that satisfies $\langle\varphi_0,\varphi_1\rangle[C] \subseteq R$. Here, $\langle\varphi_0,\varphi_1\rangle \colon C \to X^2$ denotes the map $C \ni t \mapsto (\varphi_0(t),\varphi_1(t)) \in X^2$. Assume f is strictly monotonically increasing. Then, X/R is the coequaliser of φ_0 and φ_1, and R is a closed set given by*

$$R = \langle\varphi_0,\varphi_1\rangle[C] \cup \langle\varphi_1,\varphi_0\rangle[C] \cup \{(x,x) \mid x \in X\}. \tag{4}$$

Proof. We first show (4) and that it is a closed equivalence relation. Set $A_i \coloneqq \langle\varphi_i,\varphi_{1-i}\rangle[C]$ $(i = 0,1)$ and $Q \coloneqq A_0 \cup A_1 \cup \{(x,x) \mid x \in X\}$. Note that this is the reflexive symmetric closure of A_0. Take non-trivial pairs $(x,y) \in Q$ and $(y,z) \in Q$ with $x \neq y$ and $y \neq z$ arbitrarily. These are of the form $x = \varphi_i(t)$, $y = \varphi_{1-i}(t) = \varphi_j(s)$, and $z = \varphi_{1-j}(s)$ for some $t,s \in C$ and $i,j \in \{0,1\}$. By assumption, f is an order-isomorphism onto the image. Hence, as $f(t) = g\circ\varphi_i(t) = g\circ\varphi_j(s) = f(s)$, it necessarily holds $t = s$. We get $(x,z) = (\varphi_i(t),\varphi_{1-j}(t)) \in Q$. Thus, the transitivity law of Q is shown. Therefore, Q is an equivalence relation that includes A_0. By definition of R, we have $A_0 \subseteq R \subseteq Q$. Taking the reflexive symmetric closures, we conclude $R = Q$. The images A_i $(i = 0,1)$ and the diagonal set $\{(x,x) \mid x \in X\}$ are closed since X is a Hausdorff space, from which it follows R is closed.

Second, we show that X/R is the coequaliser of φ_0 and φ_1 in $\mathbf{CCPoHaus}_{/\mathbf{R}}$. To this end, it is sufficient to show that X/R is a CC-pospace endowed with the quotient topology and order. Let $q \colon X \twoheadrightarrow Y \coloneqq X/R$ be the quotient map. Define $h \coloneqq g \circ q^{-1} \colon Y \to \mathbf{R}$. As h is strictly monotonically increasing, the quotient order on Y is antisymmetric. Take a pair of points $(q(x), q(x')) \in Y^2$ with $q(x) \not\leq q(x')$

arbitrarily. We have $x \not\lesssim x'$ by the monotonicity of q. Set $K \coloneqq \cup_{i=0,1}\varphi_i[C]$. If both x and x' belong to K, we have $q(x) \gneq q(x')$, in which case we can separate them by some open superlevel and sublevel sets of h. Hence, we assume $x \notin K$ without loss of generality. We consider the two cases: x' belongs to K or not.

Case $x' \in K$. The closed subset $K \cap \uparrow x$ is a compact (possibly empty) set that does not contain x'. Set $a \coloneqq \inf g[K \cap \uparrow x]$ (if empty, $a \coloneqq +\infty$). Since $x \lneq y$ for all $y \in K \cap \uparrow x$, we have $g(x) \lneq a$. We also have $g(x') \lneq a$. We thus take such $b \in \mathbf{R}$ that $\max\{g(x), g(x')\} \lneq b \lneq a$. Then we have $x \not\lesssim y$ for all $y \in K_{\leq b}$, where $K_{\leq b} \coloneqq K \cap g^{-1}[\downarrow b]$. For each $y \in K_{\leq b}$, we take open neighbourhoods U_y of x and V_y of y that satisfy $U_y \times V_y \subset (\not\lesssim)$. (Note that x is fixed.) As $K_{\leq b} \subset \cup_y V_y$ is an open cover of a compact set, it has a finite subcover $\cup_{i=1}^n V_{y_i}$. Set

$$U \coloneqq \left(g^{-1}(-\infty, b) \setminus K\right) \cap \bigcap_i U_{y_i}, \qquad V \coloneqq g^{-1}(-\infty, b) \cap \bigcup_i V_{y_i}. \tag{5}$$

Then, we get $(x, y) \in U \times V \subset (\not\lesssim)$. By the construction, $q[U] \times q[V] \subset (\not\leq)$ follows from $U \times K_{<b} \subset (\not\lesssim)$, where $K_{<b} \coloneqq K \cap g^{-1}(-\infty, b) = K \cap V$. As U does not intersect K, the image $q[U]$ is an open set in Y. Also $q[V]$ is an open set. This is because $q^{-1}[q[V]] = V$ holds true, which follows from $q^{-1}[q[K_{<b}]] = K_{<b}$ and $q^{-1}[q[V \setminus K_{<b}]] = V \setminus K_{<b}$. Note that the definition of the quotient map and (4) lead to these equalities.

Case $x' \notin K$. Take open neighbourhoods U of x and V of x' with $U \times V \subset (\not\lesssim)$. As $[q(x), q(x)'] = \emptyset$, the images of q of the (possibly empty) compact sets $K_{\mathrm{up}} \coloneqq K \cap \uparrow x$ and $K_{\mathrm{low}} \coloneqq K \cap \downarrow x'$ are disjoint. Consider the compact chain $q[K]$. In this chain, let u and ℓ be the minimum of $q[K_{\mathrm{up}}]$ and the maximum of $q[K_{\mathrm{low}}]$, respectively, where we promise $\min \emptyset = \max q[K]$ and $\max \emptyset = \min q[K]$ for convention. Then we have $u \gneq \ell$. Define the closed set L,

$$L \coloneqq q^{-1}[\downarrow m_{\mathrm{low}} \cup \uparrow m_{\mathrm{up}}],$$

where

$$(m_{\mathrm{up}}, m_{\mathrm{low}}) \coloneqq \begin{cases} (u, \ell) & \text{if } q[K] \cap [\ell, u] = \{\ell, u\}, \\ (m, m) & \text{for some } m \in q[K] \text{ with } \ell \lneq m \lneq u, \text{ otherwise.} \end{cases}$$

Then, L includes K in either case. We obtain open sets $U' \coloneqq U \setminus L$ and $V' \coloneqq V \setminus L$ that do not intersect K. Thus, the images $q[U']$ and $q[V']$ are open. Next, we check that $(q[U'] \times q[V']) \cap (\leq) = \emptyset$ holds. As $q[U']$ does not intersect $\downarrow m_{\mathrm{low}}$, its upset closure $\uparrow q[U']$ neither. Hence, we get $q[K] \cap \uparrow q[U'] \subseteq \uparrow m_{\mathrm{up}} \setminus \{m_{\mathrm{low}}\}$. Similarly, we get $q[K] \cap \downarrow q[V'] \subseteq \downarrow m_{\mathrm{low}} \setminus \{m_{\mathrm{up}}\}$. Combining these, we obtain $q[K] \cap \uparrow q[U'] \cap \downarrow q[V'] = \emptyset$. Suppose $(q[U'] \times q[V']) \cap (\leq) \neq \emptyset$ for contradiction. Take such $(a, b) \in U' \times V'$ that $q(a) \leq q(b)$. As $a \not\lesssim b$, there is some $(a', b') \in R \cap K^2$ with $a \leq a'$ and $b' \leq b$. However, we have $q(a') = q(b') \in q[K] \cap \uparrow q[U'] \cap \downarrow q[V']$, which is a contradiction. We thus conclude $q[U'] \times q[V'] \subset (\not\leq)$.

In summary, we have proved that the subset $(\not\leq)$ of Y^2 is an open set; namely, $Y = X/R$ is a CC-pospace. □

4 Ordered trees

Diverse definitions of trees exist, ranging from graph theory to topology and mathematical logic. However, these traditional notions do not fit well when considering metrics among Reeb posets, which can behave like "trees" under certain conditions. This section aims to unify tree definitions to accommodate various Reeb poset structures. We will first review existing tree concepts before proposing an extension. Then, we will introduce the category of locally maximal chains, crucial for differentiating "paths" in a poset. Using this, we introduce the analogy of fundamental groups for graph-like posets. Finally, we investigate a notion of ordered trees, extending the other tree notions in a certain sense.

4.1 Preliminaries on trees

Consider a connected undirected graph. A cycle is a non-empty path in which only the first and last vertices are equal, and no other vertices are duplicated. The graph is said to be a tree if and only if it contains no cycles. Note that the definition of "paths" also depends on literature and context. Readers should take care of confusing similar terms such as a walk, a trail, and even a simple path from graph theory. In this article, we devote a later subsection to the precise definition of a paths and a free category.

Let T be a tree and consider its geometric realisation $|T| \in \mathbf{Top}$. Since T contains no cycles, any closed curve in $|T|$ can be contracted; namely, $|T|$ is simply connected. Conversely, for any graph T, if $|T|$ is simply connected, T is a tree. Thus, we may alternatively define a tree as a simplicial 1-complex (an undirected graph) whose geometric realisation is simply connected. This correspondence relates a discrete tree (as a graph) and a continuous tree (as a topological space).

On the one hand, a tree is viewed as an undirected object, as explained above. On the other hand, a tree can also be regarded as an ordered object. Let $T = (V, E)$ be a tree and $r \in V$ a fixed root node. (T, r) is called a rooted tree. As T is a tree, for any two nodes x and y in T, there uniquely exists the shortest path from x to y, denoted by $p(x, y)$. All the paths $p(r, \bullet)$ from the root node can be ordered by inclusion relation; namely, we say $p(r, x) \subset p(r, y)$ if and only if $p(r, y)$ coincides with the concatenated path of $p(r, x)$ and $p(x, y)$. As the set of paths $\{p(r, x) \mid x \in V\}$ has 1-to-1 correspondence to V, the inclusion relation induces a partial order $\leq$ on V, which is called the tree order of the rooted tree (T, r). The tree order has characterisation by chains; for any node $x \in V$, its downset closure $\downarrow x$ is a chain or a totally ordered subset. The map $\downarrow x \mapsto p(r, x)$ is one-to-one. From this viewpoint, a downset can be regarded as an alternative to a path in this formulation of the tree order. Note that once a root node is fixed, the undirected rooted tree can be viewed as a directed tree. Indeed, we can conversely regard a finite poset $(V, \leq)$ with the unique minimum element r as a (directed) rooted tree $(V, \lessdot, r)$ exactly when the same property of downsets holds. Here, the symbol $\lessdot$ denotes the covering relation of $\leq$; namely, $\lessdot$ is the minimal binary relation whose reflexive transitive closure coincides with $\leq$. The binary relation $\lessdot$ is also known as a transitive reduction of $\leq$, and the directed graph $(V, \lessdot)$ as a Hasse diagram. This characterisation of a rooted tree is also used for an infinite poset in the literature of mathematical logic. In this context, the notion of the

Hasse diagram does not make sense anymore because, in general, a covering relation may not exist for an infinite poset. The Hasse diagram is also crucial in the Reeb ordering since it mathematically extracts a graph structure from a finite Reeb poset. In any case, the information of downsets gives rise to a relation between rooted trees and (finite) posets.

4.2 The category of locally maximal chains

In those well-known mathematical formulations of trees, what is essential is a path in a certain sense; a tree from graph theory is characterized by cycles, one from general topology by contractible closed curves, and a rooted tree from mathematical logic by totally ordered downsets. In the last formulation, it is significant that downsets are locally maximal chains. Generally speaking, locally maximal chains are important objects when one deals with posets. As we want to characterize a Reeb poset as a tree-like object, we focus on the viewpoint of locally maximal chains.

Let $(P, \leq)$ be a poset and C a chain in P. We say C is maximal if there is no larger chain C' such that $C \subsetneq C' \subseteq P$. Assume C has the minimum a and maximum b. We say that the chain C is locally maximal if and only if it is maximal in the closed interval $[a, b]$. For example, consider a finite directed rooted tree $(V, \lessdot, r)$. Each downset $\downarrow x$ is a chain of the form $[r, x]$. The chain is locally maximal, where the minimum is r and the maximum is x.

Significantly, any locally maximal chain in a CC-pospace is closed. To see this, note that one can express any locally maximal chain C in a CC-pospace X as $C = \bigcap_{x \in C}(\uparrow x \cup \downarrow x) \cap [\min C, \max C]$, an intersection of closed sets. Thus, a locally maximal chain could be an alternative for a directed path in a rooted tree.

Definition 14 (Category of locally maximal chains)**.** *Let $(P, \leq)$ be a poset. We define $\mathbf{MC}(P)$, the category of locally maximal chains in P, by*

$$\mathrm{Ob}\,\mathbf{MC}(P) = P, \quad \mathrm{Mor}\,\mathbf{MC}(P) = \{\textit{locally maximal chains in } P\}.$$

Here, each $C \in \mathrm{Mor}\,\mathbf{MC}(P)$ is regarded as a morphism from $\min C$ to $\max C$, and the union of locally maximal chains gives the composition.

Note that if there are some closed intervals $[a, b]$ that are not totally ordered, then we have $\#\mathrm{Hom}_{\mathbf{MC}(P)}(a, b) > 1$ on the contrary to $\#\mathrm{Hom}_P(a, b) \leq 1$. Thus, in general, there are many ways (morphisms) to "walk" from a to b in $\mathbf{MC}(P)$.

4.3 Categories of paths

Regarding locally maximal chains as "directed edges" on a poset, we will introduce "undirected paths" on the poset, in which one can go backwards from a higher point to a lower point. After that, we construct algebras of undirected paths on the poset. To this end, we use the localisation of a category.

Let $G = (V, E)$ be a directed multiple graph; V is the set of vertices, and E is the set of labelled directed edges between the vertices. For a directed edge e from x to y, its source and terminal are denoted by $s(e) = x$ and $t(e) = y$, respectively. Let $p = (v, e) \in V^{n+1} \times E^n$ $(n \in \mathbf{N})$. We say that p is a directed path from v_0 to v_n in G

if and only if

$$s(e_i) = v_i, \quad t(e_i) = v_{i+1} \qquad (i = 0, 1, \ldots, n-1).$$

It is often helpful to identify a directed path p with the following diagram in G,

$$p = (v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{n-1}} v_n \text{ in } G).$$

The free category $\mathbf{Free}(G)$ generated by G is defined by

$$\mathrm{Ob}\,\mathbf{Free}(G) = V, \quad \mathrm{Mor}\,\mathbf{Free}(G) = \{\text{directed paths in } G\}.$$

A morphism $p \colon v \to w$ in $\mathbf{Free}(G)$ is a directed path p from v to w. The composition in $\mathbf{Free}(G)$ is given by the path concatenation. For each vertex v, there uniquely exists the empty path $\varnothing_v = (v, \varnothing)$ of length 0 at v, which is the identity morphism. Note that a category $\mathbf{C}$ can be regarded as a directed multiple graph $(\mathrm{Ob}\,\mathbf{C}, \mathrm{Mor}\,\mathbf{C})$, whose free category has two morphisms $\varnothing_v$ of length 0 and (v, v, id_v) of length 1 for each object v. To avoid confusion, we need to distinguish these two, so identity morphisms in a free category are denoted by $\varnothing_v$ and not by id_v.

Definition 15 (Localisation of category)**.** *Let $\mathbf{C}$ be a category and $R \subseteq \mathrm{Mor}\,\mathbf{C}$ a subclass of morphisms. Set $G \coloneqq (\mathrm{Ob}\,\mathbf{C}, \mathrm{Mor}\,\mathbf{C} \cup R^{\mathrm{op}})$, where R^{op} denotes the subclass of opposite morphisms distinguished from any in $\mathrm{Mor}\,\mathbf{C}$. For each morphism $R \ni f \colon x \to y$, $f^{\mathrm{op}} \colon y \to x$ denotes the opposite morphism in R^{op}. The localisation $\mathbf{C}[R^{-1}]$ of $\mathbf{C}$ along R is defined by*

$$\mathrm{Ob}\,\mathbf{C}[R^{-1}] = \mathrm{Ob}\,\mathbf{C}, \quad \mathrm{Mor}\,\mathbf{C}[R^{-1}] = \mathrm{Mor}\,\mathbf{Free}(G)/{\sim},$$

where the equivalence relation $\sim$ is generated by the following rules.

(L.α) $(x \xrightarrow{f} y \xrightarrow{f^{\mathrm{op}}} x \text{ in } G) \sim (x \xrightarrow{\mathrm{id}_x} x \text{ in } G)$ *for* $R \ni f \colon x \to y$ *in* $\mathbf{C}$,

(L.β) $(x \xrightarrow{g^{\mathrm{op}}} y \xrightarrow{g} x \text{ in } G) \sim (x \xrightarrow{\mathrm{id}_x} x \text{ in } G)$ *for* $R \ni g \colon y \to x$ *in* $\mathbf{C}$,

(L.γ) $(x \xrightarrow{f} y \xrightarrow{g} z \text{ in } G) \sim (x \xrightarrow{g \circ f} z \text{ in } G)$ *for composable* $f, g \in \mathrm{Mor}\,\mathbf{C}$,

(L.δ) $(x \xrightarrow{\mathrm{id}_x} x \text{ in } G) \sim \varnothing_x$ *for* $x \in \mathrm{Ob}\,\mathbf{C}$.

We refer to the equivalence relation $\sim$ as the localisation relation over $\mathbf{Free}(G)$.

Roughly speaking, $\mathbf{C}[R^{-1}]$ is a modification of $\mathbf{C}$ so that each morphism in R is made invertible. Using the localisation, we introduce the "algebra of undirected paths" on a poset.

Definition 16 (Categories of chain paths)**.** *Let $X \in \mathbf{PoSet}$. Setting $G(X) \coloneqq (X, \mathrm{Mor}\,\mathbf{MC}(X) \cup \mathrm{Mor}\,\mathbf{MC}(X)^{\mathrm{op}})$, we define the category of chain paths of X as $\mathbf{CPath}(X) \coloneqq \mathbf{Free}(G(X))$, and the category of localised chain paths of X as $\mathbf{LPath}(X) \coloneqq \mathbf{MC}(X)[\mathrm{Mor}\,\mathbf{MC}(X)^{-1}] = \mathbf{CPath}(X)/{\sim}$. We call $p \colon x \to y$ in $\mathbf{CPath}(X)$ a chain path from x to y, and $p \colon x \to y$ in $\mathbf{LPath}(X)$ a localised chain path from x to y.*

Example 17 (Example of $\mathbf{LPath}(X)$)**.** *Let $X = \{a, b, c, d\}$ endowed with partial order $a \le b \le d$, $a \le c \le d$ where b and c are incomparable. The category of locally maximal chains, $\mathbf{MC}(X)$, is illustrated in Fig. 17. In the graph $G(X)$, there are many non-trivial paths since we have added opposite directed edges; for example,*

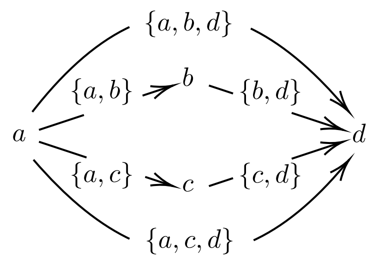


Fig. 1 The category $\mathbf{MC}(X)$

- $(b \xrightarrow{\{a,b\}^{\mathrm{op}}} a \xrightarrow{\{a,b,d\}} d)$
- $(a \xrightarrow{\{a,b\}} b \xrightarrow{\{b,d\}} d \xrightarrow{\{a,c,d\}^{\mathrm{op}}} a)$
- $(a \xrightarrow{\{a,c\}} c \xrightarrow{\{c,d\}} d \xrightarrow{\{b,d\}^{\mathrm{op}}} b \xrightarrow{\{a,b\}^{\mathrm{op}}} a)$
- $(a \xrightarrow{\mathrm{id}_a} a \xrightarrow{\{a,c,d\}} d \xrightarrow{\mathrm{id}_d} d)$

Under the localisation relation $\sim$, we can simplify some of the above examples.

- $(b \xrightarrow{\{a,b\}^{\mathrm{op}}} a \xrightarrow{\{a,b,d\}} d) \sim (b \xrightarrow{\{b,d\}} d)$
- $(a \xrightarrow{\{a,b\}} b \xrightarrow{\{b,d\}} d \xrightarrow{\{a,c,d\}^{\mathrm{op}}} a) \sim (a \xrightarrow{\{a,b,d\}} d \xrightarrow{\{a,c,d\}^{\mathrm{op}}} a)$
- $(a \xrightarrow{\mathrm{id}_a} a \xrightarrow{\{a,c,d\}} d \xrightarrow{\mathrm{id}_d} d) \sim (a \xrightarrow{\{a,c,d\}} d)$

Thus, by localising morphisms, we can obtain the category of localised chain paths where chain paths can be reduced by the identity, composition and inversion laws; similar routes are identified. Consider $\mathrm{Hom}_{\mathbf{LPath}(X)}(a, a)$*, which consists of "closed paths" from a to itself. Only the "simple" closed paths are of the forms* $p := (a \to b \to d \to c \to a)$ *or its opposite* p^{op}*. Since* p *and* p^{op} *are inverse of each other in* $\mathbf{LPath}(X)$*, we have*

$$\mathrm{Hom}_{\mathbf{LPath}(X)}(a, a) = \{p^n \mid n \in \mathbf{Z}\},$$

which is group isomorphic to $\mathbf{Z}$*.*

As we see in Example 17, the category of localised chain paths is considered to reflect the geometric property of a poset viewed as an undirected graph or Hasse diagram. In other words, $\mathbf{LPath}(X)$ can be regarded as a first homotopy algebra. From this viewpoint, we introduce the following notion.

Definition 18 (Fundamental group of poset)**.** *Let* $X \in \mathbf{PoSet}$*. The fundamental group of* X *at a base point* $a \in X$ *is defined by*

$$\pi_1(X; a) := \mathrm{Hom}_{\mathbf{LPath}(X)}(a, a).$$

Since all endomorphisms in $\mathbf{LPath}(X)$ are invertible by definition, obviously $\pi_1(X; a)$ is a group with the composition operation.

Now, by analogy with graphs, we can introduce the notion of a partially ordered tree.

Definition 19 (Partially ordered tree)**.** *Let* $X \in \mathbf{PoSet}$ *be* $\lesseqgtr$*-connected, where* $(\lesseqgtr) := (\leq) \cup (\geq)$ *is the comparability relation in* X*. We say that* X *is a partially ordered tree if and only if the fundamental groups of* X *at any base points are trivial;* $\pi_1(X; a) \cong \{0\}$ *for all* $a \in X$*.*

In other words, we can regard a $\lesseqgtr$-connected poset as a tree exactly when it is "simply connected." Indeed, we can confirm that this definition generalises similar notions via the following proposition.

Proposition 20. *Let X be a finite $\lesseqgtr$-connected poset. Then, the following are equivalent.*

(i) *X is a partially ordered tree.*

(ii) *$\#\mathrm{Hom}_{\mathbf{LPath}(X)}(a, b) = 1$ for all $a, b \in X$.*

(iii) *The Hasse diagram of X is a tree in the sense of graph theory; namely, there are no cycles.*

(iv) *The geometric realisation of the Hasse diagram of X is simply connected; namely, any loop can be contracted to a single point in the realisation.*

Proof. (ii) $\Longrightarrow$ (i) and (iii) $\Longleftrightarrow$ (iv) are trivial. (i) $\Longrightarrow$ (iii) is also trivial by contraposition. It suffices to show (iii) $\Longrightarrow$ (ii). We can construct the following 1-to-1 correspondence

$$\mathrm{Mor}\,\mathbf{LPath}(X) \to \{\text{“canonical” chain paths in } \mathrm{Mor}\,\mathbf{CPath}(X)\},$$

which takes a representative chain path from a localised chain path, an equivalence class. See Section 5 for the construction. (iii) indicates that any "canonical" chain path from a to b can be identified with the shortest path from a to b. Once these correspondences are admitted, we obtain $\#\mathrm{Hom}_{\mathbf{LPath}(X)}(a, b) = 1$. □

Although the construction of the 1-to-1 correspondence seems intuitively clear under the assumption of finiteness on the poset, in general, such a canonical map does not always exist. In the next section, we will precisely state the reducibility of chain paths and the correspondence between the equivalence classes and the representative chain paths.

5 Branch structures of posets

In this section, we reveal how to simplify chain paths from the viewpoint of localised chain paths. As a localised chain path is an equivalence class, we often need its good canonical representative element. Indeed, under certain assumptions, any chain path can be rewritten to a canonical one. Intuitively, we can always obtain the most straightforward or shortest path by applying an appropriate homotopy. In a branch of theoretical computer science, such a property is called normalisation, and an object is said to be in normal form if it can no longer be rewritten or simplified. The theorem we prove in this section is a strong normalisation theorem found in computer science. Under certain assumptions on a poset X, any chain path $p \in \mathrm{Mor}\,\mathbf{CPath}(X)$ has its unique normal form p' obtained by a reduction sequence from p to p'. Hence, we borrow some terms, notions, and even proof techniques from computer science to prove this theorem. We first look at branch structures of a poset to regard it as a graph-like object. We then consider a reduction system for chain paths, demonstrating its local confluence. This local behaviour ensures the unique existence of a normal form for each chain path, evidenced by global confluence.

5.1 Branch completeness

Let us consider the category $\mathbf{MC}(X)$ of locally maximal chains in a poset X. Take a span in $\mathbf{MC}(X)$,

$$S = \left(x \xleftarrow{f} s \xrightarrow{g} y \right).$$

Since chains f and g have the minimum s, their intersection is non-empty. Although $f \cap g$ is also a chain, it is not always locally maximal. In order to consider branch structure, the notion of simultaneous factorisation is useful.

Definition 21 (Final branch of span and initial branch of cospan)**.** *Let* $\mathbf{C}$ *be a category and consider two spans in* $\mathbf{C}$*,*

$$S = \left(x \xleftarrow{f} s \xrightarrow{g} y \right), \quad T = \left(x \xleftarrow{f'} t \xrightarrow{g'} y \right).$$

We say that a morphism $h\colon s \to t$ *in* $\mathbf{C}$ *is a simultaneous factorisation from* S *to* T *if* $f = f' \circ h$ *and* $g = g' \circ h$ *hold. Let* $B = (x \leftarrow b \to y)$ *be a span in* $\mathbf{C}$ *and* k *a simultaneous factorisation from* S *to* B*. We call* (B, k) *a final branch of* S *if and only if, for all simultaneous factorisation* h *from* S *to* T*, there uniquely exists a simultaneous factorisation* ℓ *from* T *to* B *such that* $\ell \circ h = k$*. When it is evident from the context, we refer to just* b *as the final branch of* S *as well.*

In a similar manner by duality, we define the initial branch B *of a cospan* $C = (x \xrightarrow{f} s \xleftarrow{g} y)$*.*

Example 22 (Branches in a poset)**.** *Let* $P \in \mathbf{PoSet}$ *and* $S = (x \leftarrow s \to y)$ *be a span in the category* P*. Suppose the infimum* $\inf\{x, y\}$ *exists. In that case, it is a final branch of* S*. Indeed, simultaneous factorisations from* S *coincide with lower bounds* $\geq s$ *of* $\{x, y\}$*, and the infimum is, by definition,* $\inf\{x, y\} = \max\{t \mid t \leq x, t \leq y\} = \max\{t \mid s \leq t, t \leq x, t \leq y\}$*. Conversely, if the final branch* b *of* S *exists, it is an infimum of* $\{x, y\}$ *taken in the subset* $\uparrow s \cap (\downarrow x \cup \downarrow y)$*. Note that, even if the final branch* b *of* S *exists, there might not be a unique infimum of* $\{x, y\}$ *taken in* X *in general. Similarly, the initial branch of a cospan* $(x \to s \leftarrow y)$ *relates to the supremum* $\sup\{x, y\}$*. Thus, the branches are generalisations of the infimum and supremum.*

Example 23 (Branches in the category of locally maximal chains)**.** *Set*

$$X \coloneqq \operatorname{colim}_{\mathbf{PoSet}}([0,1]_a \leftarrow \{0,1\} \to [0,1]_b),$$

where the morphisms are inclusions and the subscripts a, b *are just labels to distinguish the two intervals. The final branch of a span in* $\mathbf{MC}(X)$*,*

$$S = \left(1 \xleftarrow{[0,1]_a} 0 \xrightarrow{[0,1]_b} 1 \right),$$

is given by 0*.*

Lemma 24 (Uniqueness of branches of factored spans)**.** *Let* $S = (x \leftarrow s \to y)$ *be a span in* $\mathbf{C}$ *and* b *the final branch of* S*. Take a span* $T = (x' \leftarrow s \to y')$ *in* $\mathbf{C}$ *such that* $x \leftarrow b$ *is factored as* $x \leftarrow x' \leftarrow b$ *and* $b \to y$ *as* $b \to y' \to y$*. Then,* b *is also the final branch of* T*.*

Proof. It is trivial by the fact that $k\colon s \to t$ factors S if and only if it factors T. $\square$

Definition 25 (Branch completeness). *We say* $\mathbf{C}$ *is branch complete if and only if it has final branches and initial branches; namely, for all spans* S*, there uniquely exists the final branch of* S *up to isomorphism, and for all cospans* C*, there uniquely exists the initial branch of* C *up to isomorphism.*

Example 26 (Non-trivial branches and branch completeness). *We revisit the poset* X *given in* (2)*, Example 12. For all non-trivial spans* $(x_a \leftarrow s \to y_b)$ *in* X *with* $x_a \geq 1_a$ *and* $y_b \geq 1_b$*, its final branch does not exist. This is because the set of lower bounds of* x_a *and* y_b *is an open interval* $(0, 1_a)$ *(or equivalently,* $(0, 1_b)$*), which does not admit of the maximum. Hence,* X *is not branch complete. Note that* $X/\{1_a \sim 1_b\}$ *becomes branch complete.*

Definition 27 (Notation of branches). *Let* $\mathbf{C}$ *be a branch complete category. For a span* $S = (x \xleftarrow{f} s \xrightarrow{g} y)$*, let* $x\ {}_f\sqcap_g\ y$ *denote the final branch of* S*. Similarly, for a cospan* $C = (x \xrightarrow{f} s \xleftarrow{g} y)$*, let* $x\ {}_f\sqcup_g\ y$ *denote the initial branch of* C*. If it is obvious which morphisms* f *and* g *are referred to in the context, we may omit them and write* $x \sqcap y$ *or* $x \sqcup y$*.*

Let us examine when $\mathbf{MC}(X)$ becomes branch complete, as shown in Lemma 29. Before showing it, we introduce an excellent property of CC-pospaces provided by the following lemma. Indeed, several similar results have already been known in some classical studies on the theory of ordered sets. For example, see Theorem 20, Chapter X in Birkhoff (1940). We demonstrate this in the context of CC-pospaces to ensure a self-contained exposition.

Lemma 28. *Let* $X \in \mathbf{CCPoHaus}$*. Assume* X *is compact and totally ordered. Then,* X *admits of the minimum and maximum.*

Proof. Consider a net $(x)_{x \in X}$. Since X is a compact Hausdorff space, a subnet of the net exists that converges to some unique element $\top \in X$. Then, $(x)_{x\in X}$ also converges to this $\top$ as X itself indexes it. Since $(\leq)$ is closed, the limit preserves the order in X. Hence, for any $a \in X$, considering the subnet $(y)_{y \in \uparrow a}$, we have $\top = \lim_{y \in \uparrow a} y \geq a$. In conclusion, $\top$ is the maximum in X. Similarly, taking the limit $(x)_{x \in X^{\mathrm{op}}}$, we can construct the minimum value. $\square$

The following lemma provides a sufficient condition for a CC-pospace to possess branch completeness regarding locally maximal chains.

Lemma 29 (Branch completeness induced by local compactness of chains). *Let* $X \in \mathbf{CCPoHaus}$*. Assume that every locally maximal chain in* X *is compact. Then,* $\mathbf{MC}(X)$ *is branch complete.*

Proof. Take a span $S = (x \xleftarrow{f} s \xrightarrow{g} y)$ in $\mathbf{MC}(X)$ arbitrarily. We will construct a final branch of S. If $f \subseteq g$ or $g \subseteq f$, then the final branch of S is trivial. We may thus assume $f \not\subseteq g$ and $g \not\subseteq f$. Consider

$$k := f \cap g \cap \bigcap_{t \in (f \setminus g) \cup (g \setminus f)} \downarrow t,$$

which is a non-empty compact chain. By Lemma 28, the maximum $b := \max k$ exists. As f and g are locally maximal, k is too. Set

$$B := \left(x \xleftarrow{f'} b \xrightarrow{g'} y \right) \quad \text{where} \quad f' := f \cap {\uparrow} b \text{ and } g' := g \cap {\uparrow} b.$$

Then, $k\colon s \to b$ in $\mathbf{MC}(X)$ is a simultaneous factorisation from S to B. The mapping by this construction $S \mapsto \mathcal{B}(S) := B$ leads to the universality of the final branch. Indeed, for any simultaneous factorisation from S to T, we have $\mathcal{B}(S) = \mathcal{B}(T) = B$. In conclusion, (B, k) is the final branch of S. As for the initial branches, the proof follows dually similarly. □

Example 30 (Non-trivial branch completeness in a CC-pospace)**.** *Set*

$$X := \mathrm{colim}_{\mathbf{CCPoHaus}} \left\{ \mathbf{R}_a \leftarrow \left[\frac{1}{2n+1}, \frac{1}{2n} \right] \to \mathbf{R}_b \right\}_{n \in \mathbf{Z}_{>0}}.$$

Here, a and b are labelled to distinguish the copies of real numbers, and the morphisms are inclusions. Infinitely many branch structures are found in X in the neighbourhood of $0 \in X$. However, as X satisfies the condition of Lemma 29, $\mathbf{MC}(X)$ is branch complete. Set $f = [0,1]_a \colon 0 \to 1_a$ and $g = [0,1]_b \colon 0 \to 1_b$. Let us follow the proof of Lemma 29. The symmetric difference of f and g consists of infinitely many real numbers of labels a and b in the neighbourhood of 0 but does not contain 0 itself. Thus, k must coincide with $\{0\}$, the final branch of f and g; $1_a\ {}_f{\sqcap}_g\ 1_b = 0$ in $\mathbf{MC}(X)$. However, 0 is not an infimum (namely, a final branch) of any pair of points in X. Removing this point, $X \setminus \{0\}$ is still branch complete, but $\mathbf{MC}(X \setminus \{0\})$ is not.

As this example illustrates, $\mathbf{MC}(X)$ does not inherit the branch completeness from the underlying poset X in general. Of course, this comes from the presence of infinitely many branch structures. It does not matter for a poset with a finite number of branch structures.

Remind that, by Lemma 13, the coequaliser of two compact chains in a CC-pospace is explicitly realised as a quotient space. We use this lemma to show that the coequaliser preserves the compactness of locally maximal chains.

Lemma 31 (Coequaliser preserves compactness of locally maximal chains)**.** *Let $(C, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ be a totally ordered compact CC-pospace. Let $(X, g) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ be a CC-pospace with a strictly monotonically increasing height function g. Let $\varphi_0, \varphi_1 \colon (C, f) \to (X, g)$ be parallel morphisms in $\mathbf{CCPoHaus}_{/\mathbf{R}}$ such that the images of φ_0 and φ_1 are incomparable; $\varphi_0[C] \times \varphi_1[C] \subseteq X^2 \setminus (\lesseqgtr)$. Assume that every locally maximal chain in X is compact. Then, the coequaliser of φ_0 and φ_1 also satisfies the same property.*

Proof. Let Y be the coequaliser of φ_0 and φ_1 as realised in Lemma 13. Let $q\colon X \to Y$ be the quotient map. Set $K := \varphi_0[C] \cup \varphi_1[C]$. Take $h\colon a \to b$ arbitrarily in $\mathbf{MC}(Y)$.

Note that, for all $x \in X$, the inverse image of q of $q(x)$ is either of the following form,

$$q^{-1}[q(x)] = \begin{cases} \{x\} & (x \notin K), \\ \{\varphi_0(t), \varphi_1(t)\} & (x \in K,\ t \coloneqq f^{-1}(g(x))). \end{cases}$$

Hence, it is sufficient to confirm that $q^{-1}[h]$ is compact. If h does not intersect with $q[K]$, $q^{-1}[h]$ is a locally maximal chain and hence is compact by assumption. We may thus assume that h intersects with $q[K]$ and $a, b \in q[K]$ without loss of generality. By assumption $\varphi_0[C] \times \varphi_1[C] \subseteq X^2 \setminus (\lesseqgtr)$, there exists such a pair (k_0, k_1) of locally maximal chains in X that $q^{-1}[h] \subset k_0 \cup k_1$, $k_i \cap \varphi_i[C] \neq \emptyset$, and $k_i \cap \varphi_{1-i}[C] = \emptyset$ $(i = 0, 1)$. (We take maximal chains in $q^{-1}[[a, b]]$ that include $q^{-1}[h] \cap \varphi_i[C]$ for $i = 0, 1$, respectively. Note that, in general, the equality $q^{-1}[h] = k_0 \cup k_1$ does not need to hold. Without the assumption of incomparability, the inverse image can not be covered only by two locally maximal chains.) Thus, $q^{-1}[h]$ is compact as it is a closed set included in a compact set $k_0 \cup k_1$. □

Remark 32 (A good colimit enjoys branch completeness)**.** *Suppose we want to construct a CC-pospace using a colimit of a certain finite diagram consisting of totally ordered CC-pospaces. In such a scenario, we often coequalise two incomparable chains in the coproduct space. Hence, the incomparability condition $\varphi_0[C] \times \varphi_1[C] \subseteq X^2 \setminus (\lesseqgtr)$ of Lemma 31 is satisfied. Furthermore, as a coequaliser preserves the compactness of locally maximal chains, a finite colimit does too. Combining Lemma 29, such a colimit enjoys branch completeness of locally maximal chains. We will use the above lemmas in such a way later.*

5.2 Reduction transform

We can construct a 1-to-1 correspondence between a span and a chain path of the following form,

$$\left(x \xleftarrow{f} s \xrightarrow{g} y\right) \text{ in } \mathbf{MC}(X) \quad \xleftrightarrow{\text{1-to-1}} \quad \left(x \xrightarrow{f^{\mathrm{op}}} s \xrightarrow{g} y\right) \text{ in } G(X).$$

Thus, if $\mathbf{MC}(X)$ is branch complete, we can introduce the following transform in $\mathbf{CPath}(X)$ using branches,

$$\left(x \xrightarrow{f^{\mathrm{op}}} s \xrightarrow{g} y\right) \longmapsto \left(x \xrightarrow{f'^{\mathrm{op}}} x \sqcap y \xrightarrow{g'} y\right).$$

Let $(x \xleftarrow{f} s \xrightarrow{g} y)$ be a span, and $(x \xleftarrow{f'} x \sqcap y \xrightarrow{g'} y)$ the factored span. Since $f' \colon x \sqcap y \to x$ is a subset of f, as chains, it can be represented uniquely as $f' = f \cap {\uparrow}(x \sqcap y)$. Similarly, we also have $g' = g \cap {\uparrow}(x \sqcap y)$. Using these properties in $\mathbf{MC}(X)$, we can represent the factored morphisms after taking the branches of (co)spans. Then, we define a reduction transform as follows.

Definition 33 (Reduction of chain paths)**.** *Assume* $\mathbf{MC}(X)$ *is branch complete. We introduce a (labelled) binary relation* $\Rightarrow$ *over* $\operatorname{Mor}\mathbf{CPath}(X)$ *that is closed under the composition operation and satisfies the following rules.*

(R.α) $(x \xrightarrow{f^{\mathrm{op}}} y \xrightarrow{g} z) \overset{\alpha}{\Longrightarrow} (x \xrightarrow{(f\cap\uparrow y')^{\mathrm{op}}} y' \xrightarrow{g\cap\uparrow y'} z)$ *for a span* $(x \xleftarrow{f} y \xrightarrow{g} z)$ *and* $y' = x \sqcap z$

(R.β) $(x \xrightarrow{f} y \xrightarrow{g^{\mathrm{op}}} z) \overset{\beta}{\Longrightarrow} (x \xrightarrow{f\cap\downarrow y'} y' \xrightarrow{(g\cap\downarrow y')^{\mathrm{op}}} z)$ *for a cospan* $(x \xrightarrow{f} y \xleftarrow{g} z)$ *and* $y' = x \sqcup z$

(R.γ) $(x \xrightarrow{f} y \xrightarrow{g} z) \overset{\gamma}{\Longrightarrow} (x \xrightarrow{g\circ f} z)$, $(z \xrightarrow{g^{\mathrm{op}}} y \xrightarrow{f^{\mathrm{op}}} x) \overset{\gamma}{\Longrightarrow} (z \xrightarrow{f^{\mathrm{op}}\circ g^{\mathrm{op}}} x)$ *for composable* $f, g \in \operatorname{Mor}\mathbf{MC}(X)$

(R.δ) $(x \xrightarrow{\mathrm{id}_x} x) \overset{\delta}{\Longrightarrow} \varnothing_x$, $(x \xrightarrow{\mathrm{id}_x^{\mathrm{op}}} x) \overset{\delta}{\Longrightarrow} \varnothing_x$ *for* $\mathrm{id}_x \in \operatorname{Mor}\mathbf{MC}(X)$

In the above rules, the labels $\tau \in \{\alpha, \beta, \gamma, \delta\}$ *are put to clarify which rule is applied. Each labelled binary relation* $\overset{\tau}{\Longrightarrow}$ *is assumed to be closed under the composition as well; namely, for all* $p, p' \in \operatorname{Mor}\mathbf{CPath}(X)$, $p \overset{\tau}{\Longrightarrow} p'$ *holds if and only if, for some* p_i, p'_i $(i = 0, 1, 2)$, $p = p_2 \circ p_1 \circ p_0$, $p' = p'_2 \circ p'_1 \circ p'_0$, *and* $p_1 \overset{\tau}{\Longrightarrow} p'_1$. *Without any labels, it means* $(\Rightarrow) = \cup_{\tau\in\{\alpha,\beta,\gamma,\delta\}}(\overset{\tau}{\Longrightarrow})$. *For each label* $\tau \in \{\alpha, \beta, \gamma, \delta\}$, *we call* $\overset{\tau}{\Longrightarrow}$ *the* τ*-reduction relation and* $\Rightarrow$ *the reduction relation. We say that* p *is* τ*-reduced to* p' *exactly when* $p \overset{\tau}{\Longrightarrow} p'$. *Furthermore, let* $\overset{*}{\Longrightarrow}$ *denote the reflexive transitive closure of* $\Rightarrow$.

Definition 34 (Normal form)**.** *A chain path* $p \in \operatorname{Mor}\mathbf{CPath}(X)$ *is said to be in normal form if and only if, for all* $q \in \operatorname{Mor}\mathbf{CPath}(X)$, *we have that* $p \overset{*}{\Longrightarrow} q$ *implies* $p = q$. *In other words,* p *is in normal form if it does not admit of further non-trivial reduction.*

Mathematicians often refer to such a notion as canonical form or irreducibility but its usage depends on the context. As we stated at the beginning of this section, we instead use the term "normal form" from computer science.

To prove the strong normalisation theorem, we confirm how the localisation relation $\sim$ and the reduction $\Rightarrow$ are related. The following lemma demonstrates the relationship, and at the same time, immediately, it leads to a property known as local confluence in computer science. In fact, the idea of the lemma derives from the Church–Rosser theorem regarding confluent rewriting systems.

Lemma 35 (Localisation and reduction)**.** *Assume* $\mathbf{MC}(X)$ *is branch complete. Let* $\sim$ *be the localisation relation over* $\mathbf{CPath}(X)$, *and* $\Rightarrow$ *the reduction relation over* $\mathbf{CPath}(X)$. *Then, for all* $p_0, p_1 \in \operatorname{Mor}\mathbf{CPath}(X)$,

(i) $p_0 \Rightarrow p_1$ *implies* $p_0 \sim p_1$, *and*

(ii) $p_0 \sim p_1$ *implies* $p_i \overset{*}{\Longrightarrow} p$ $(i = 0, 1)$ *for some* $p \in \operatorname{Mor}\mathbf{CPath}(X)$.

Proof. We first note that the generation rules (L.α)–(L.δ) of localisation are corresponding with those (R.α)–(R.δ) of reduction, respectively. Especially, the implication

(i) for the labels γ and δ is trivial. It suffices to show (i) for α-reduction. Assume that

$$p_0 = (x \xrightarrow{f^{\mathrm{op}}} y \xrightarrow{g} z) \overset{\alpha}{\Longrightarrow} (x \xrightarrow{f^{\mathrm{op}} \cap \uparrow y'} y' \xrightarrow{g \cap \uparrow y'} z) = p_1,$$

where $y' = x \sqcap z$. Applying the rule (L.γ) followed by (L.δ), we get

$$\begin{aligned} p_0 &\sim \left(x \xrightarrow{f^{\mathrm{op}} \cap \uparrow y'} y' \xrightarrow{f^{\mathrm{op}} \cap \downarrow y'} y \xrightarrow{g \cap \downarrow y'} y' \xrightarrow{g \cap \uparrow y'} z \right) \\ &\sim \left(x \xrightarrow{f^{\mathrm{op}} \cap \uparrow y'} y' \xrightarrow{g \cap \uparrow y'} z \right) = p_1. \end{aligned}$$

Here, (L.δ) can be applied since the chain $f \cap \downarrow y'$ coincides with $g \cap \downarrow y'$.

Second, we show (ii). By induction on the localisation relation, it suffices to show (ii) for the rules (L.α)–(L.δ). The cases (L.γ) and (L.δ) are trivial. Assume (L.α) holds; namely,

$$p_0 = (x \xrightarrow{f^{\mathrm{op}}} y \xrightarrow{f} x) \sim (x \xrightarrow{\mathrm{id}_x} x) = p_1.$$

Then, applying α-reduction followed by δ-reduction, we get

$$p_0 \overset{\alpha}{\Longrightarrow} \left(x \xrightarrow{\{x\}^{\mathrm{op}}} x \;{}_{f^{\mathrm{op}}}\!\sqcap_f\, x \xrightarrow{\{x\}} x \right) = \left(x \xrightarrow{\mathrm{id}_x^{\mathrm{op}}} x \xrightarrow{\mathrm{id}_x} x \right) \overset{\delta}{\Longrightarrow} \left(x \xrightarrow{\mathrm{id}_x} x \right) = p_1.$$

Thus, setting $p = p_1$, we have $p_i \overset{*}{\Longrightarrow} p$ for $i = 0, 1$. The case (L.β) follows similarly. $\square$

On the one hand, as we see in Lemma 35, localisation and reduction are similar concepts by which we can simplify chain paths. On the other hand, the valuable property of the reduction is that specific chain paths can be reduced to their normal forms, as shown in the following lemma.

Lemma 36 (Zig-zag reduction)**.** *Let p be a chain path of length 3 of the following form,*

$$p = \left(p_0 \xrightarrow{f} p_1 \xrightarrow{g^{\mathrm{op}}} p_2 \xrightarrow{h} p_3 \right).$$

Then, possible non-trivial reduction sequences from p keeping length 3 are only of the following forms,

$$\begin{aligned} p &\overset{\alpha}{\Longrightarrow} \left(p_0 \xrightarrow{f} p_1 \xrightarrow{g_\alpha^{\mathrm{op}}} p_1 \sqcap p_3 \xrightarrow{h_\alpha} p_3 \right) \\ &\overset{\beta}{\Longrightarrow} \left(p_0 \xrightarrow{f_\beta} p_0 \sqcup (p_1 \sqcap p_3) \xrightarrow{g_{\alpha\beta}^{\mathrm{op}}} p_1 \sqcap p_3 \xrightarrow{h_\alpha} p_3 \right) =: p^{\alpha\beta}, \\ \textit{or} \quad p &\overset{\beta}{\Longrightarrow} \left(p_0 \xrightarrow{f'_\beta} p_0 \sqcup p_2 \xrightarrow{g_\beta^{\mathrm{op}}} p_2 \xrightarrow{h} p_3 \right) \\ &\overset{\alpha}{\Longrightarrow} \left(p_0 \xrightarrow{f'_\beta} p_0 \sqcup p_2 \xrightarrow{g_{\beta\alpha}^{\mathrm{op}}} (p_0 \sqcup p_2) \sqcap p_3 \xrightarrow{h'_\alpha} p_3 \right) =: p^{\beta\alpha}. \end{aligned}$$

Furthermore, if the reduced chain path, either $p^{\alpha\beta}$ or $p^{\beta\alpha}$, contains no identity morphisms, then it is in normal form and coincides with another.

Proof. First, we briefly check about $p \overset{*}{\Rightarrow} p^{\alpha\beta}$ and $p \overset{*}{\Rightarrow} p^{\beta\alpha}$. The length of a chain path is shortened by γ or δ-reductions. Thus, only α or β-reductions are applicable to keep the length. In addition, non-trivial α-reduction can be applied at most once by the uniqueness of final branches. Therefore, a non-trivial reduction applicable following α-reduction while keeping the length 3 is β-reduction only. Similarly, α-reduction following β-reduction vice versa is also possible.

Second, we see $p^{\alpha\beta}$ without identity morphisms is in normal form. Assume

$$p_0^{\alpha\beta} \lneq p_1^{\alpha\beta}, \quad p_1^{\alpha\beta} \gneq p_2^{\alpha\beta}, \quad p_2^{\alpha\beta} \lneq p_3^{\alpha\beta},$$

where $p_i^{\alpha\beta}$ denotes the i-th object in the chain path $p^{\alpha\beta}$ for $i = 0, 1, 2, 3$. We can not apply γ or δ-reduction to $p^{\alpha\beta}$. Thus, $p^{\alpha\beta} \overset{\alpha}{\Rightarrow} p^{\alpha\beta}$ is the rest to be checked, which indeed follows from Lemma 24.

Finally, we prove $p^{\alpha\beta}$ containing no identity morphisms implies $p^{\alpha\beta} = p^{\beta\alpha}$. Consider the locally maximal chain g as a subset of interval $[p_2, p_1]$. As both $p_0 \sqcup p_2$ and $p_1 \sqcap p_3$ belong to the same chain, g, they are comparable. Suppose $p_0 \sqcup p_2 \leq p_1 \sqcap p_3$ for contradiction. Then, we have a locally maximal chain $g \cap [p_0 \sqcup p_2, p_1 \sqcap p_3]$, which is $p_0 \sqcup p_2 \to p_1 \sqcap p_3$ in $\mathbf{MC}(X)$. We thus obtain $p_0 \to p_0 \sqcup p_2 \to p_1 \sqcap p_3$, from which it follows $p_1^{\alpha\beta} = p_0 \sqcup (p_1 \sqcap p_3) = p_1 \sqcap p_3 = p_2^{\alpha\beta}$. However, this means $p_1^{\alpha\beta} \to p_2^{\alpha\beta}$ is identity, which is a contradiction. Hence, $p_0 \sqcup p_2 \gneq p_1 \sqcap p_3$ holds. We have a locally maximal chain $g \cap [p_1 \sqcap p_3, p_0 \sqcup p_2]$, which is $p_1 \sqcap p_3 \to p_0 \sqcup p_2$ in $\mathbf{MC}(X)$. Applying Lemma 24 to the spans $(p_1 \leftarrow p_1 \sqcap p_3 \to p_3)$ and $(p_0 \sqcup p_2 \leftarrow p_1 \sqcap p_3 \to p_3)$, we obtain $p_1 \sqcap p_3 = (p_0 \sqcup p_2) \sqcap p_3$. Hence, $p_2^{\alpha\beta} = p_2^{\beta\alpha}$ holds, and $p_1^{\alpha\beta} = p_1^{\beta\alpha}$ also follows similarly from Lemma 24 for cospans by duality. As p_0 and p_3 are unchanged via the reductions, we conclude $p^{\alpha\beta} = p^{\beta\alpha}$. □

5.3 Strong normalisation

Using the properties from Lemma 35 and 36 for "small-steps" reductions, we can show the uniqueness of normal forms for "big-steps" reductions, which behave as a function.

Theorem 37 (Strong normalisation)**.** *Assume $\mathbf{MC}(X)$ is branch complete. Then, for all chain paths $p \in \mathrm{Mor}\,\mathbf{CPath}(X)$, there uniquely exists p' in normal form such that $p \overset{*}{\Rightarrow} p'$.*

Proof. We first see the uniqueness. Let p'_i be a chain path in normal form obtained by $p \overset{*}{\Rightarrow} p'_i$ for each $i = 0, 1$. From Lemma 35, it is necessary that $p'_0 \sim p'_1$ holds true; hence, there exists some p' such that $p'_i \overset{*}{\Rightarrow} p'$ $(i = 0, 1)$. Since both p'_0 and p'_1 are in normal form, we get $p'_0 = p'_1 = p'$.

It suffices to show the existence of normal form. Let $n(\bullet) \in \mathbf{N}$ denote the length of a chain path. Then, the length $n(\bullet)$ is monotonically decreasing within a reduction sequence, whose minimum length exists,

$$m = \min\left\{ n(p') \,\middle|\, p \overset{*}{\Rightarrow} p' \right\}.$$

Let p' be a chain path obtained by a reduction sequence from p that attains the minimum length m. Since we can not apply δ-reduction anymore to p', there are no identity morphisms in p'. If $m \leq 1$, it is trivial that p' is in the normal form. Assume $m \geq 2$. Let p'_i denote the i-th object in the chain path p'. We may assume $p'_0 \lneq p'_1$ without loss of generality. By minimumness, we can not apply γ-reduction to p', so it necessarily holds $p'_1 \gneq p'_2$. Repeating this deduction, we get $p'_{2i} \lneq p'_{2i+1}$ and $p'_{2i-1} \gneq p'_{2i}$. We consider any sequence from p' which consists only of α or β-reductions. We are applying Lemma 36 to every subpath of length 3 in the sequence, which results in the unique normal form via at most $m-1$ reductions. □

Definition 38 (ω-reduction)**.** *Define the binary relation over* $\operatorname{Mor} \mathbf{CPath}(X)$*,*

$$(\overset{\omega}{\Longrightarrow}) := \left\{ (p, p') \in (\overset{*}{\Rightarrow}) \,\middle|\, p' \textit{ is in normal form} \right\}.$$

Corollary 39. *Assume* $\mathbf{MC}(X)$ *is branch complete. Then,* $\overset{\omega}{\Longrightarrow}$ *is a functional relation and, for all* $p_0, p_1 \in \operatorname{Mor} \mathbf{MC}(X)$*,* $p_0 \sim p_1$ *implies* $p_0 \overset{\omega}{\Longrightarrow} p$ *and* $p_1 \overset{\omega}{\Longrightarrow} p$ *for some* p *in normal form.*

Recall $\mathbf{LPath}(X) = \mathbf{CPath}(X)/\sim$. The corollary states that $\overset{\omega}{\Longrightarrow}$ induces a map from $\operatorname{Mor} \mathbf{LPath}(X)$ to the set of chain paths in normal forms. Hence, if $\mathbf{MC}(X)$ is branch complete, we can canonically take a representative chain path in normal form from any equivalence class $p \in \operatorname{Mor} \mathbf{LPath}(X)$.

6 Interleaving distance between ordered tree spaces

In this section, we introduce interleaving distance between partially ordered trees. The construction of the interleaving distance is almost the same as one proposed by de Silva et al. (2016). However, to this end, we need an alternative smoothing endofunctor over $\mathbf{CCPoHaus}_{/\mathbf{R}}$. In order to prove that such a functor preserves the structure of partially ordered trees, we utilise the strong normalisation theorem.

6.1 Interleaving distance between Reeb graphs in $\mathbf{Top}_{/\mathbf{R}}$

The stability of Reeb graphs has been proved by V. de Silva, E. Munch, *et al.* using the interleaving distance in the framework of the slice category $\mathbf{Top}_{/\mathbf{R}}$. We briefly review their work.

An object $(X, f) \in \mathbf{Top}_{/\mathbf{R}}$ is called an $\mathbf{R}$-space. Let $\tilde{\mathcal{R}}(X, f)$ denote the Reeb graph of $(X, f) \in \mathbf{Top}_{/\mathbf{R}}$; namely, $\tilde{\mathcal{R}}(X, f) = X/\sim$ where the equivalence relation $\sim$ is defined as in Section 2. As the quotient induces a continuous function $\tilde{f} \colon \tilde{\mathcal{R}}(X, f) \to \mathbf{R}$, the Reeb graph $\tilde{\mathcal{R}}(X, f)$ becomes an $\mathbf{R}$-space. $\tilde{\mathcal{R}}$ is a functor $\mathbf{Top}_{/\mathbf{R}} \to \mathbf{Top}_{/\mathbf{R}}$.

We then define a smoothing functor $\tilde{\mathcal{U}}_\varepsilon \colon \mathbf{Top}_{/\mathbf{R}} \to \mathbf{Top}_{/\mathbf{R}}$ as $\tilde{\mathcal{U}}_\varepsilon(X, f) = \tilde{\mathcal{R}}(X \times [-\varepsilon, \varepsilon], f_\varepsilon)$ with $f_\varepsilon \colon X \times [-\varepsilon, \varepsilon] \ni (x, t) \mapsto f(x) + t \in \mathbf{R}$ for any $\varepsilon \geq 0$. The family of smoothing functors $(\tilde{\mathcal{U}}_\varepsilon)_{\varepsilon \geq 0}$ forms a semigroup of endofunctors on the category of Reeb graphs; namely, we have $\tilde{\mathcal{U}}_0 \simeq \mathrm{Id}$ and $\tilde{\mathcal{U}}_{\varepsilon_1}\tilde{\mathcal{U}}_{\varepsilon_2} \simeq \tilde{\mathcal{U}}_{\varepsilon_1+\varepsilon_2}$ for all $\varepsilon_1, \varepsilon_2 \geq 0$. Let $\varepsilon \geq 0$ and $(X, f), (Y, g) \in \mathbf{Top}_{/\mathbf{R}}$. Let $\tilde{X}$ and $\tilde{Y}$ denote the Reeb graphs $\tilde{\mathcal{R}}(X, f)$ and $\tilde{\mathcal{R}}(Y, g)$, respectively. Two morphisms $\varphi \colon \tilde{X} \to \tilde{\mathcal{U}}_\varepsilon \tilde{Y}$ and $\psi \colon \tilde{Y} \to \tilde{\mathcal{U}}_\varepsilon \tilde{X}$ are called ε-isomorphisms between $\tilde{X}$ and $\tilde{Y}$ if and only if the following diagrams commute.

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{[\![\bullet,0]\!]} & \tilde{\mathcal{U}}_\varepsilon \tilde{X} & \xrightarrow{[\![\bullet,0]\!]} & \tilde{\mathcal{U}}_{2\varepsilon} \tilde{X} \\ & \searrow^{\varphi} & & \nearrow_{\tilde{\mathcal{U}}_\varepsilon \psi} & \\ & & \tilde{\mathcal{U}}_\varepsilon \tilde{Y} & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \tilde{\mathcal{U}}_\varepsilon \tilde{X} & & \\ & \nearrow^{\psi} & & \searrow^{\tilde{\mathcal{U}}_\varepsilon \varphi} & \\ \tilde{Y} & \xrightarrow{[\![\bullet,0]\!]} & \tilde{\mathcal{U}}_\varepsilon \tilde{Y} & \xrightarrow{[\![\bullet,0]\!]} & \tilde{\mathcal{U}}_{2\varepsilon} \tilde{Y} \end{array} .$$

Here, for $Z \in \mathbf{Top}_{/\mathbf{R}}$ and $\varepsilon \geq 0$, $[\![\bullet, \bullet]\!] = [\![\bullet, \bullet]\!]_{\tilde{\mathcal{U}}_\varepsilon Z} \colon Z \times [-\varepsilon, \varepsilon] \to \tilde{\mathcal{U}}_\varepsilon Z$ denotes the quotient map associated with $\tilde{\mathcal{R}}$. We now define the interleaving distance between Reeb graphs, also known as the Reeb distance, as follows.

$$d_{\mathrm{R}}(\tilde{X}, \tilde{Y}) \coloneqq \inf\left\{\varepsilon \geq 0 \,\middle|\, \text{there exist } \varepsilon\text{-isomorphisms between } \tilde{X} \text{ and } \tilde{Y}\right\}.$$

Under certain conditions on constructibility of $\mathbf{R}$-spaces $(X, f), (X, g) \in \mathbf{Top}_{/\mathbf{R}}$, V. de Silva, E. Munch, *et al.* have proved the stability theorem,

$$d_{\mathrm{R}}(\tilde{\mathcal{R}}(X, f), \tilde{\mathcal{R}}(X, g)) \leq \|f - g\|_\infty.$$

The proof is easy. Assume $\varepsilon \coloneqq \|f - g\|_\infty < \infty$. Define

$$\begin{aligned} &\varphi \colon \tilde{\mathcal{R}}(X, f) \ni [x] \mapsto [\![x, f(x) - g(x)]\!] \in \tilde{\mathcal{U}}_\varepsilon \tilde{\mathcal{R}}(X, g) \text{ and} \\ &\psi \colon \tilde{\mathcal{R}}(X, g) \ni [x] \mapsto [\![x, g(x) - f(x)]\!] \in \tilde{\mathcal{U}}_\varepsilon \tilde{\mathcal{R}}(X, f). \end{aligned}$$

As $|f(x) - g(x)| \leq \varepsilon$ for all $x \in X$, the morphisms φ and ψ are well-defined. By definition, it follows $\tilde{\mathcal{U}}_\varepsilon \psi \circ \varphi = [\![\bullet, 0]\!]_{\tilde{\mathcal{U}}_{2\varepsilon}\tilde{\mathcal{R}}(X,f)}$. Thus, φ and ψ are ε-isomorphisms.

Note that although we used a different domain and codomain of the functors than in the original paper to save paper space, it still keeps the essential idea of the interleaving distance and the stability theorem. Rigorously speaking, as done by V. de Silva, E. Munch, *et al.*, we should regard d_{R} as an extended metric; namely, it may take positive infinite values in general.

6.2 Interleaving distance between Reeb posets in $\mathbf{CCPoHaus}_{/\mathbf{R}}$

To generalise the stability result to the case of Reeb posets, we need modification to the distance function and the proof of the stability theorem. First, the definition of a smoothing functor matters when it comes to the discrete setting of input data. One advantage of the Reeb ordering method is the ability to deal with general input data structures, including meshes and grids. On the one hand, in the case of simplicial triangulation meshes, the resulting Reeb graphs are known to be stable, as shown by de Silva et al. (2016). On the other hand, in the case of grid-type structures, we can not interpolate input data uniquely to realise a continuous scalar function. We often use the planar lattice graph with eight-point adjacencies at each grid point to reduce the number of topological noises in the practical application of the Reeb ordering method. In such cases, the resulting Reeb poset should be regarded as a poset with discrete topology. Hence, we formulate a smoothing functor in the over category of CC-pospaces over $\mathbf{R}$. Second, we need some technical workaround to show the well-definedness of ε-isomorphisms. To this end, we establish some lemmas regarding partially ordered trees to ensure that a smoothing functor preserves the tree structures.

6.2.1 Finite subdiagram and its colimit

When we take the colimit of diagrams in **CCPoHaus**, we need some good conditions to keep the underlying set structure. In Example 26, we have shown that the colimit of a diagram in **CCPoHaus** differs from that in **PoSet**. The presence of non-compactness caused this difference from the viewpoint of Lemma 13; if two paths were compact, the lemma could have coequalised them without any concern about the underlying set structure. Thus, as long as we consider situations to which we can apply Lemma 13, there are no problems taking colimit in **CCPoHaus**. However, there is another problem in the case infinitely many morphisms exist in a diagram, where infinitely many coequalisers may be needed to construct the colimit. We can give a similar example just as Example 26, where non-compactness becomes troublesome again, even if we are coequalising only compact paths. Therefore, to clarify the underlying set of the colimit in **CCPoHaus**, we need a particular notion of finiteness.

Let X be a poset. We define

$$\mathbf{Ch}(X) \coloneqq \left\{c \in \mathbf{2}^X \;\middle|\; c \text{ is a non-empty chain in } X\right\} \text{ and}$$
$$\mathbf{I}(X) \coloneqq \left\{c \cap c' \in \mathbf{2}^X \;\middle|\; c, c' \in \mathbf{Ch}(X) \text{ are maximal chains with } c \cap c' \neq \emptyset\right\},$$

where $\mathbf{2}^X$ denotes the power set of X. We regard $\left(\mathbf{2}^X, \subseteq\right)$ as a category (or equivalently just as a poset) by inclusion relations.

Lemma 40. *Let $\mathcal{D}\colon \mathbf{Ch}(X) \to \mathbf{C}$ be a diagram in a category $\mathbf{C}$. Then, the isomorphism $\operatorname{colim} \mathcal{D} \cong \operatorname{colim}(\mathcal{D}\mathcal{I})$ holds, where $\mathcal{I}\colon \mathbf{I}(X) \hookrightarrow \mathbf{Ch}(X)$ is the inclusion functor.*

To show this, we use the well-known fact from category theory: a functor is final if and only if it preserves the colimits of any given diagram when composed from the right. See Fact 65 in Appendix A for the details.

Proof of Lemma 40. We apply Fact 65 to the inclusion functor $\mathcal{I}$ to show the desired isomorphism. Thus, we check the connectedness of $\mathcal{I}$-under categories. Take any $c \in \mathbf{Ch}(X)$. The category ${}^{c/}\mathcal{I}$ of objects $\mathcal{I}$-under c is explicitly written as

$$\operatorname{Ob} {}^{c/}\mathcal{I} = \{i \in \mathbf{I}(X) \mid c \subseteq i\}, \quad \operatorname{Mor} {}^{c/}\mathcal{I} = \left\{i \subseteq j \;\middle|\; i, j \in \operatorname{Ob} {}^{c/}\mathcal{I}\right\}.$$

Take $i, j \in {}^{c/}\mathcal{I}$ arbitrarily. By definition, there exist $x, y, z, w \in \mathbf{Ch}(X)$ such that $x \cap y = i$, $z \cap w = j$, and $c \subseteq x \cap y \cap z \cap w$. Obviously, $k \coloneqq y \cap z$ $(\supseteq c)$ is non-empty, and we thus have $k \in {}^{c/}\mathcal{I}$. Consequently, we get a zig-zag path $i \subseteq y \supseteq k \subseteq z \supseteq j$ in ${}^{c/}\mathcal{I}$ between arbitrary i and j, implying that the under category ${}^{c/}\mathcal{I}$ is connected. □

We introduce the following two notions.

Definition 41 (Chain finite)**.** *Let X be a poset. We say X is chain finite if and only if $\#\operatorname{Ob}\mathbf{I}(X) < \infty$.*

Definition 42 (Chain final topology)**.** *Let $X \in$ **CCPoHaus**. We say that the topology of X is chain final if and only if the following isomorphism holds,*

$$X \cong \operatorname{colim}_{\mathbf{CCPoHaus}} \mathcal{J},$$

*where $\mathcal{J} \colon \mathbf{Ch}(X) \hookrightarrow$ **CCPoHaus** is an inclusion functor regarding $\mathbf{Ch}(X)$ as a subcategory of **CCPoHaus** by subspace topologies.*

Example 43. *Let $X \in$ **CCPoHaus** be a finite poset endowed with the discrete topology. Then, X is chain finite, and the topology of X is chain final.*

Example 44. *Consider an Euclidean space $\mathbf{R}^n$ endowed with the product order of the standard orders for $n \geq 2$. Then, $\mathbf{R}^n$ is not chain finite, and the standard topology of $\mathbf{R}^n$ is not chain final. To see this, consider, for $a < b$ in $\mathbf{R}^n$, an open interval (a, b). It is not open in the standard topology but must be open in the chain final topology because we have $(a, b) = \cup_{C \in \mathbf{Ch}(\mathbf{R}^n)} (a, b) \cap C$.*

We immediately obtained the following corollary.

Corollary 45. *Let X be a chain finite CC-pospace endowed with chain final topology. Then, X can be reconstructed by taking a colimit of a finite diagram $\mathcal{J}\mathcal{I}$; namely, $X \cong \operatorname{colim}_{\mathbf{CCPoHaus}}(\mathcal{J}\mathcal{I})$.*

Indeed, the above corollary highly relates to the preceding study by Wolk (1958), where it has been shown that an order-compatible topology in a poset is uniquely determined if the poset is of finite width. If a poset is chain finite, it is of finite width necessarily. (Note that the converse is not true as the poset in Example 30 is of width two but is not chain finite.) Thus, the assumption that the topology is chain final in Corollary 45 seems redundant. Also, chain finiteness is a too strong assumption compared to width finiteness. We leave this gap problem open and will focus on our application hereafter.

6.2.2 Smoothing functor in $\mathbf{CCPoHaus}_{/\mathbf{R}}$

For $X = (X, \mathcal{O}_X, \leq, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$, set $X_{\pm\varepsilon} \coloneqq X \times [-\varepsilon, \varepsilon]$. We introduce the following pair of an $\mathbf{R}$-valued function f and a binary relation R on the product space $X_{\pm\varepsilon}$.

$$f_{\pm\varepsilon} \colon X_{\pm\varepsilon} \to \mathbf{R}; \qquad f_{\pm\varepsilon}(x, t) = f(x) + t,$$
$$R = \left\{ ((x, t), (y, s)) \in X_{\pm\varepsilon}^2 \;\middle|\; x \lesseqgtr y \right\}.$$

Let us consider the $f_{\pm\varepsilon}$-climbing quasi-order $\lesssim_{f_{\pm\varepsilon}\uparrow}$ on $(X_{\pm\varepsilon}, R)$. This correspondence from X to $X_{\pm\varepsilon}$ forms a functor denoted by $\mathcal{E}_\varepsilon \colon \mathbf{CCPoHaus}_{/\mathbf{R}} \to \mathbf{QoTop}_{/\mathbf{R}}$; for any

$\varphi\colon X \to Y$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$,

$$\mathcal{E}_\varepsilon X \coloneqq \big(X_{\pm\varepsilon}, \mathcal{O}_X \times \mathcal{O}_{[-\varepsilon,\varepsilon]}, \lesssim_{f_{\pm\varepsilon}\uparrow}, f_{\pm\varepsilon}\big),$$
$$\mathcal{E}_\varepsilon \coloneqq \varphi \times \mathrm{id}_{[-\varepsilon,\varepsilon]}\colon \mathcal{E}_\varepsilon X \to \mathcal{E}_\varepsilon Y.$$

Let $\mathcal{P}\colon \mathbf{QoTop}_{/\mathbf{R}} \to \mathbf{CCPoHaus}_{/\mathbf{R}}$ denote the CC-pospace reflector endowed with $\mathbf{R}$-valued functions. We define

$$\mathcal{U}_\varepsilon \coloneqq \mathcal{P}\mathcal{E}_\varepsilon\colon \mathbf{CCPoHaus}_{/\mathbf{R}} \to \mathbf{CCPoHaus}_{/\mathbf{R}}, \tag{6}$$

which we call an ε-smoothing functor.

Definition 46 (Smoothed CC-pospace)**.** *For $X \in \mathbf{CCPoHaus}_{/\mathbf{R}}$, we call $\mathcal{U}_\varepsilon X$ an ε-smoothed CC-pospace of X. The quotient map inducing the ε-smoothed CC-pospace from the product space $X_{\pm\varepsilon}$ is denoted by $[\![\bullet,\bullet]\!]\colon X_{\pm\varepsilon} \to \mathcal{U}_\varepsilon X$.*

Note that for any $(X, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$, we can regard the height function f itself as a morphism $f\colon (X, f) \to (\mathbf{R}, \mathrm{id})$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$. Conversely, f is the unique morphism in $\mathbf{CCPoHaus}_{/\mathbf{R}}$ of such type $(X, f) \to (\mathbf{R}, \mathrm{id})$. Hence, the morphism $\mathcal{U}_\varepsilon f\colon \mathcal{U}_\varepsilon X \to \mathcal{U}_\varepsilon \mathbf{R} \cong (\mathbf{R}, \mathrm{id})$ coincides with the associated height function of the ε-smoothed CC-pospace $\mathcal{U}_\varepsilon X$. Hereafter, for convention, we always identify $\mathcal{U}_\varepsilon \mathbf{R}$ with the Euclidean line $\mathbf{R}$ for all $\varepsilon \geq 0$. Indeed, we can express $\mathcal{U}_\varepsilon f$ simply as follows by definition.

Proposition 47. *We have $\mathcal{U}_\varepsilon f([\![x,t]\!]) = f_{\pm\varepsilon}(x,t)$ for all $(x,t) \in X_{\pm\varepsilon}$.*

In addition, except for the case $\varepsilon = 0$, these height functions are strictly order-preserving by definition of the climbing order.

Proposition 48. *Let $(X, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$. For all $\varepsilon > 0$, the height function $\mathcal{U}_\varepsilon f$ is a strictly order-preserving map that preserves the strict orders; namely, for all $\xi, \eta \in \mathcal{U}_\varepsilon X$, if $\xi < \eta$, it holds $\mathcal{U}_\varepsilon f(\xi) < \mathcal{U}_\varepsilon f(\eta)$.*

Note that there maybe other definitions for a map to be strictly order-preserving but we use the above definition as was done by Schröder (2016). As was shown for the smoothing functors in $\mathbf{Top}_{/\mathbf{R}}$, the smoothing functors $(\mathcal{U}_\varepsilon)_{\varepsilon\geq 0}$ also form a semigroup.

Proposition 49. *Let $(X, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$. Then, the following hold.*

1. *If f is strictly order-preserving, then $\sigma_0 \coloneqq [\![\bullet, 0]\!]\colon (X, f) \to \mathcal{U}_0(X, f)$ is an isomorphism in $\mathbf{CCPoHaus}_{/\mathbf{R}}$.*
2. *For all $a, b \geq 0$, $\sigma_{a,b}\colon \mathcal{U}_a\mathcal{U}_b(X, f) \to \mathcal{U}_{a+b}(X, f)$ is an isomorphism in $\mathbf{CCPoHaus}_{/\mathbf{R}}$, where $\sigma_{a,b}([\![[\![x,t]\!], s]\!]) \coloneqq [\![x, t+s]\!]$.*

Proof. First, we show $\mathcal{U}_0 X \cong X$. Assume f is strictly order-preserving. Obviously, $(X_{\pm 0}, \leq_{f_{\pm 0}\uparrow})$ is isomorphic to $(X, \leq_{f\uparrow})$ where $\leq_{f\uparrow}$ is the climbing order on $(X, \lesseqgtr)$. By definition $(\leq) \subseteq (\leq_{f\uparrow})$ is trivial. We check the converse. By the assumption and the contraposition, for all pairs $(x,y) \in X^2$ with $x \lesseqgtr y$, $f(x) \geq f(y)$ implies $x \geq y$. Let $x, y \in X$ with $x \leq_{f\uparrow} y$. Suppose the case $x \lesseqgtr y$. Then, $f(x) \leq f(y)$ holds by definition, from which it follows $x \leq y$. Suppose the general case where x and y are incomparable. Then, we take a path from x to y on the compatibility graph $(X, \lesseqgtr)$ along which f is increasing. Applying the same proof to the sequence pairwisely, we get $x \leq y$ by transitivity. Thus, we obtain $(\leq_{f\uparrow}) \subseteq (\leq)$. Hence, these two orders coincide, and $\leq_{f\uparrow}$

is a partial order. We get X itself by taking the CC-pospace reflection of $(X, \leq_{f\uparrow})$, concluding $\mathcal{U}_0 X \cong X$.

Second, we confirm that $\mathcal{E}_a\mathcal{E}_b X$ coincides with $X_{a,b}$ in $\mathbf{QoTop}_{/\mathbf{R}}$ for any $a, b \geq 0$, where

$$X_{a,b} \coloneqq \left(X \times [-b,b] \times [-a,a], \lesssim_{f_{a,b}\uparrow}\right), \qquad f_{a,b}(x,t,s) \coloneqq f(x) + t + s.$$

Here, we identify the underlying set of $\mathcal{E}_a\mathcal{E}_b X$ with that of $X_{a,b}$, as both are product spaces. Set R and S as follows.

$$\begin{aligned}
R_{a,b} &\coloneqq \left\{((x,t,s),(x',t',s')) \in X_{a,b}^2 \;\middle|\; f_{a,b}(x,t,s) \leq f_{a,b}(x',t',s')\right\},\\
R &\coloneqq R_{a,b} \cap \left\{((x,t,s),(x',t',s')) \in X_{a,b}^2 \;\middle|\; x \lesseqgtr x'\right\},\\
S &\coloneqq R_{a,b} \cap \left\{((x,t,s),(x',t',s')) \in X_{a,b}^2 \;\middle|\; \begin{array}{l}(x,t) \lesssim_{f_{\pm b}\uparrow} (x',t')\\ \text{or } (x',t') \lesssim_{f_{\pm b}\uparrow} (x,t)\end{array}\right\}.
\end{aligned}$$

These two relations, R and S, generate $\lesssim_{f_{a,b}\uparrow}$ and $\lesssim_{(f_{\pm b})_{\pm a}\uparrow}$ by the transitive closures, respectively. Here, the quasi-order on $\mathcal{E}_a\mathcal{E}_b X$ is denoted by $\lesssim_{(f_{\pm b})_{\pm a}\uparrow}$, or S^*. By definition, $R \subseteq S$ is trivial. Hence, we show $S \subseteq R^*$. Take $((x,t,s),(x',t',s')) \in S$ arbitrarily. By definition of the climbing order, there is some path $((x_n,t_n))_{n=0}^N$ from (x,t) to (x',t') on $X \times [-b,b]$ such that $x_n \lesseqgtr x_{n+1}$ holds for all n, and $(\theta_n)_{n=0}^N$ is monotonically increasing or decreasing, where $\theta_n \coloneqq f_{\pm b}(x_n,t_n)$. For real numbers $A, B \in \mathbf{R}$, define $I_A^B \colon [0,1] \to [A,B]$ as $I_A^B(u) \coloneqq (1-u)A + uB$. The map I_A^B is an order-preserving (resp. order-reversing) bijection if and only if $A < B$ (resp. $A > B$) holds. Set $A \coloneqq f_{\pm b}(x,t)$ and $B \coloneqq f_{\pm b}(x',t')$. If $A = B$, then $\theta_n = A$ holds for all n and we get an increasing sequence $(f_{a,b}(x_n,t_n,s_n))$ for any increasing sequence $(s_n)_{n=0}^N$ from s to s'. Thus, $(x,t,s)\ R^*\ (x',t',s')$ holds trivially true if $A = B$. We assume $A \neq B$, where I_A^B is a bijection. Set $s_n \coloneqq I_s^{s'} \circ (I_A^B)^{-1} \circ f_{a,b}(x_n,t_n)$. For each n, as $\min\{s,s'\} \leq s_n \leq \max\{s,s'\}$, we have $s_n \in [-a,a]$. Furthermore, we get

$$\begin{aligned}
f_{a,b}(x_n,t_n,s_n) &= \left(I_A^B + I_s^{s'}\right) \circ (I_A^B)^{-1}(\theta_n)\\
&= I_{A+s}^{B+s'} \circ (I_A^B)^{-1}(\theta_n).
\end{aligned}$$

Since $A + s = f_{a,b}(x,t,s) \leq f_{a,b}(x',t',s') = B + s'$, the map $I_{A+s}^{B+s'} \circ (I_A^B)^{-1}$ is order-preserving (resp. order-reversing). Hence, $(f_{a,b}(x_n,t_n,s_n))_n$ is increasing, implying $(x,t,s)\ R^*\ (x',t',s')$. We thus conclude $S \subseteq R^*$, from which it follows $R^* = S^*$; namely, $X_{a,b} = \mathcal{E}_a\mathcal{E}_b X$.

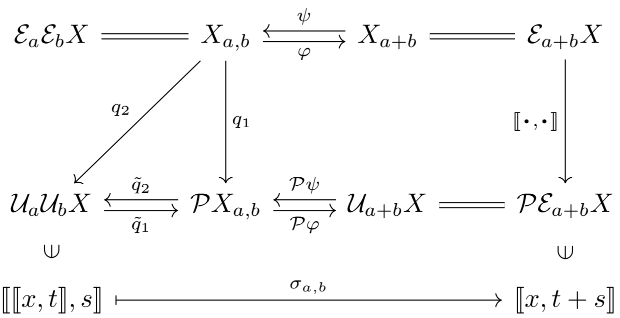
Third, we show $\mathcal{P}X_{a,b} \cong \mathcal{U}_{a+b}X$ for any $a, b \geq 0$. Set $X_{a+b} \coloneqq \mathcal{E}_{a+b}X$. Define morphisms between $X_{a,b}$ and X_{a+b} as follows.

$$\begin{aligned}
&\varphi \colon X_{a,b} \to X_{a+b}; \quad \varphi(x,t,s) = (x, t+s),\\
&\psi \colon X_{a+b} \to X_{a,b}; \quad \psi(x,u) = \left(x, \frac{bu}{a+b}, \frac{au}{a+b}\right).
\end{aligned}$$

Applying $\mathcal{P}$, we get $\mathcal{P}\varphi\colon \mathcal{P}X_{a,b} \to \mathcal{P}X_{a+b} = \mathcal{U}_{a+b}X$. Clearly, we have $\varphi \circ \psi = \mathrm{id}_{X_{a+b}}$, which induces $P\varphi \circ \mathcal{P}\psi = \mathrm{id}_{\mathcal{P}X_{a+b}}$. Take $(x, s, t) \in X_{a,b}$ arbitrarily. We have

$$\psi \circ \varphi(x, s, t) = \left(x, \frac{b}{a+b}(t+s), \frac{a}{a+b}(t+s)\right).$$

Obviously, (x, s, t) and $\psi \circ \varphi(x, s, t)$ are equivalent in $X_{a,b}$ by definition of the climbing order, implying that $\mathcal{P}\psi \circ \mathcal{P}\varphi = \mathrm{id}_{\mathcal{P}X_{a,b}}$. Hence, the inverse of $\mathcal{P}\varphi$ is $\mathcal{P}\psi$, which implies $\mathcal{P}X_{a,b} \cong \mathcal{U}_{a+b}X$ holds in $\mathbf{CCPoHaus}_{/\mathbf{R}}$.



in $\mathbf{QoTop}_{/\mathbf{R}}$.

Finally, we confirm that $\mathcal{U}_a\mathcal{U}_bX$ is isomorphic to $\mathcal{P}X_{a,b}$. Let $q_1\colon X_{a,b} \twoheadrightarrow \mathcal{P}X_{a,b}$ denote the quotient map. Let $q_2\colon X_{a,b} \twoheadrightarrow \mathcal{U}_a\mathcal{U}_bX$ be the map defined by $q_2(x,t,s) = [\![[\![x,t]\!],s]\!]$. Both q_1 and q_2 are order-preserving continuous maps to CC-pospaces. By universality of the CC-pospace reflection, there is a unique morphism $\tilde{q}_2\colon \mathcal{P}X_{a,b} \to \mathcal{U}_a\mathcal{U}_bX$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$ that factors as $q_2 = \tilde{q}_2 \circ q_1$. Similarly, as $\mathcal{U}_a = \mathcal{P}\mathcal{E}_a$, there is a unique morphism $\tilde{q}_1\colon \mathcal{U}_a\mathcal{U}_bX \to \mathcal{P}X_{a,b}$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$ that factors as $q_1 = \tilde{q}_1 \circ q_2$. Hence, we get $q_1 = \tilde{q}_1 \circ \tilde{q}_2 \circ q_1$ and $q_2 = \tilde{q}_2 \circ \tilde{q}_1 \circ q_2$. From these equalities and universality of the CC-pospace reflection again, it follows that $\mathrm{id} = \tilde{q}_1 \circ \tilde{q}_2$ and $\mathrm{id} = \tilde{q}_2 \circ \tilde{q}_1$. We thus have $\mathcal{U}_a\mathcal{U}_bX \cong \mathcal{P}X_{a,b}$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$. Note that the desired isomorphism from $\mathcal{U}_a\mathcal{U}_bX$ to $\mathcal{U}_{a+b}X$ is given by $\sigma_{a,b} = \mathcal{P}\varphi \circ \tilde{q}_1$, which completes the proof. □

6.2.3 Smoothing functor represented by chains of real values

The above formulation of a smoothing functor is helpful at the point that the universality induces $[\![\bullet,\bullet]\!]\colon X_{\pm\varepsilon} \to \mathcal{U}_\varepsilon X$. However, when keeping the structure of the chains in mind, the presence of the product space $X_{\pm\varepsilon}$ tends to introduce a certain level of difficulty. Therefore, we introduce another formulation of a smoothing functor to deal with the order structure intuitively.

Lemma 50. *$\mathcal{U}_\varepsilon$ preserves colimits.*

Proof. $\mathcal{P}$ preserves colimits, and $\mathcal{E}_\varepsilon$ does coproducts. Thus, it suffices to show that $\mathcal{E}_\varepsilon$ preserves coequalisers. Take arbitrarily a pair of parallel morphisms $\varphi, \psi\colon (X, \lesssim_X, f_X) \to (Y, \lesssim_Y, f_Y)$ in $\mathbf{QoTop}_{/\mathbf{R}}$. Let $q\colon Y \twoheadrightarrow Z$ be the coequaliser of φ and ψ, and $Q\colon \mathcal{E}_\varepsilon Y \twoheadrightarrow \bar{Z}$ that of $\mathcal{E}_\varepsilon\varphi$ and $\mathcal{E}_\varepsilon\psi$. By universality of the coequalisers, there is a well-defined bijection $Q(y,t) \mapsto \mathcal{E}_\varepsilon q(y,t)$. We check that this bijection and its inverse are both order-preserving.

Remind that the following climbing orders on graphs characterise both quasi-orders on $\bar{Z}$ and $\mathcal{E}_\varepsilon Z$.

$$Q(y,t) \lesssim_{\bar{Z}} Q(y',t') \iff (y,t) \lesssim_{Q\uparrow} (y',t') \text{ on } (\mathcal{E}_\varepsilon Y, (\lesssim_{\mathcal{E}_\varepsilon Y}) \cup (\sim_Q)),$$
$$\mathcal{E}_\varepsilon q(y,t) \lesssim_{\mathcal{E}_\varepsilon Z} \mathcal{E}_\varepsilon q(y',t') \iff (q(y),t) \lesssim_{f_{Z,\pm\varepsilon}\uparrow} (q(y'),t') \text{ on } \mathcal{E}_\varepsilon Z.$$

Here, $\sim_Q$ denotes the equivalence relation on $\mathcal{E}_\varepsilon Y$ by the quotient map Q. Take $(y,t) \lesssim_{Q\uparrow} (y',t')$ arbitrarily such that $(y,t) \lesssim_{\mathcal{E}_\varepsilon Y} (y',t')$ or $(y,t) \sim_Q (y',t')$. It is clear that the latter case immediately implies $(q(y),t) = (q(y'),t')$ and hence $\mathcal{E}_\varepsilon q(y,t) \lesssim_{\mathcal{E}_\varepsilon Z} \mathcal{E}_\varepsilon q(y',t')$. Assume the former $(y,t) \lesssim_{\mathcal{E}_\varepsilon Y} (y',t')$ holds. By definition of the climbing order on $\mathcal{E}_\varepsilon Y$, there are some path $(y_i)_{i=0}^n$ from y to y' on the graph $(Y, (\lesssim_Y) \cup (\gtrsim_Y))$ and sequence $(t_i)_{i=0}^n$ on $[-\varepsilon,\varepsilon]$ from t to t' such that $(f_{Y,\pm\varepsilon}(y_i,t_i))$ is increasing. Applying q to the path, we get a path $(q(y_i))$ on $(Z, (\lesssim_Z) \cup (\gtrsim_Z))$ from $q(y)$ to $q(y')$ with $(f_{Z,\pm\varepsilon}(q(y_i),t_i))$ increasing. Therefore, we obtain $\mathcal{E}_\varepsilon q(y,t) \lesssim_{\mathcal{E}_\varepsilon Z} \mathcal{E}_\varepsilon q(y',t')$.

Take $\mathcal{E}_\varepsilon q(y,t) \lesssim_{\mathcal{E}_\varepsilon Z} \mathcal{E}_\varepsilon q(y',t')$ arbitrarily such that $q(y)$ and $q(y')$ are comparable and $f_{Z,\pm\varepsilon}(q(y),t) \le f_{Z,\pm\varepsilon}(q(y'),t')$. Suppose $q(y) \gtrsim_Z q(y')$. Setting $a \coloneqq f_Y(y)$, $b \coloneqq f_Y(y')$, $A \coloneqq a+t$ and $B \coloneqq b+t'$, we get $t \le t'$, $a \ge b$ and $A \le B$. As $q(y) \gtrsim_Z q(y')$, there is some path $(y_i)_{i=0}^n$ from y' to y on the graph $(Y, (\lesssim_Y) \cup (\sim_q))$ with $(q(y_i))$ increasing. Take a decreasing sequence $(t_i)_{i=0}^n$ from t' to t in $[-\varepsilon,\varepsilon]$ such that, for all $i<n$, $f_{Y,\pm\varepsilon}(y_i,t_i) \ge f_{Y,\pm\varepsilon}(y_{i+1},t_{i+1})$ and that, for all $i<n$, $y_i \sim_q y_{i+1}$ implies $t_i = t_{i+1}$. (To construct such a sequence explicitly, see the similar proof in Proposition 49 using I_A^B. We must take care of a pair at which $y_i \sim_q y_{i+1}$ holds.) Then, we have, for all i,

$$\begin{cases} Q(y_i,t_i) = Q(y_{i+1},t_{i+1}) & \text{if } y_i \sim_q y_{i+1}, \\ (y_i,t_i) \gtrsim_{\mathcal{E}_\varepsilon Y} (y_{i+1},t_{i+1}) & \text{otherwise.} \end{cases}$$

Thus, $((y_{n-i},t_{n-i}))_{i=0}^n$ becomes a path from (y,t) to (y',t') on the graph $(\mathcal{E}_\varepsilon Y, (\lesssim_{\mathcal{E}_\varepsilon Y}) \cup (\sim_Q))$; namely, $(y,t) \lesssim_{Q\uparrow} (y',t')$. Suppose the case $q(y) \lesssim_Z q(y')$. Constructing a path realising the Q-climbing order $(y,t) \lesssim_{Q\uparrow} (y',t')$ is similarly straightforward. In either case, we conclude $Q(y,t) \lesssim_{\bar{Z}} Q(y',t')$.

Finally, by considering the transitivity law, we conclude that both $\mathcal{E}_\varepsilon Z \to \bar{Z}$ and $\bar{Z} \to \mathcal{E}_\varepsilon Z$ are order-preserving, completing the proof. ☐

Lemma 51. *Let $(X, \mathcal{O}_X, \le, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ be endowed with a chain final topology. Then, for any $\varepsilon \ge 0$, there is such a diagram $\mathcal{D}_\varepsilon \colon \mathbf{Ch}(X) \to \mathbf{CCPoHaus}_{/\mathbf{R}}$ that all objects of $\mathcal{D}_\varepsilon$ are totally ordered and that $\mathcal{U}_\varepsilon X \cong \operatorname{colim} \mathcal{D}_\varepsilon$. Furthermore, if $\varepsilon > 0$, all the objects of $\mathcal{D}_\varepsilon$ can be regarded as subspaces of $\mathbf{R}$.*

Proof. Let $\mathcal{J} \colon \mathbf{Ch}(X) \to \mathbf{CCPoHaus}_{/\mathbf{R}}$ be the inclusion functor. X has a chain final topology, so we have $X \cong \operatorname{colim} \mathcal{J}$. Hence, applying Lemma 50, we get $\mathcal{U}_\varepsilon X \cong \mathcal{U}_\varepsilon \operatorname{colim} \mathcal{J} \cong \operatorname{colim} \mathcal{U}_\varepsilon \mathcal{J}$. Therefore, the diagram $\mathcal{U}_\varepsilon \mathcal{J}$ is what we wanted. Assume $\varepsilon > 0$. We check that each object of the diagram is a subspace of $\mathbf{R}$. Take $C \in \mathbf{Ch}(X)$ arbitrarily. Then, we have

$$\mathcal{U}_\varepsilon \mathcal{J} C = \mathcal{P}\mathcal{E}_\varepsilon C \cong (C \times [-\varepsilon,\varepsilon])/{\sim} \cong f[C] + [-\varepsilon,\varepsilon].$$

Here, $\sim$ denotes the equivalence relation associated with the quasi-order on $\mathcal{E}_\varepsilon C$, which must satisfy

$$(x, t) \sim (y, s) \iff f(x) + t = f(y) + s,$$

from which the last isomorphism follows. $\square$

Theorem 52 (Branch completeness of smoothed CC-pospaces)**.** *Let* $(X, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ *be a chain finite CC-pospace endowed with a chain final topology and a strictly monotonically increasing height function. Assume that every locally maximal chain in* X *is compact. Then,* $\mathbf{MC}(\mathcal{U}_\varepsilon(X, f))$ *is branch complete for all* $\varepsilon \geq 0$*.*

Proof. The proof is straightforward from Lemmas 29, 31, 40, and 51. $\square$

6.2.4 Interleaving distance

We now define the interleaving distance between Reeb graphs. Let $(X, f), (Y, g) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ and $\varepsilon \geq 0$. Let $\varphi\colon X \to \mathcal{U}_\varepsilon Y$ and $\psi\colon Y \to \mathcal{U}_\varepsilon X$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$ and consider the following diagrams.

$$\begin{array}{ccccc} X & \xrightarrow{[\![\bullet,0]\!]} & \mathcal{U}_\varepsilon X & \xrightarrow{[\![\bullet,0]\!]} & \mathcal{U}_{2\varepsilon} X \\ & \searrow^{\varphi} & & \nearrow_{\mathcal{U}_\varepsilon \psi} & \\ & & \mathcal{U}_\varepsilon Y & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \mathcal{U}_\varepsilon X & & \\ & \nearrow^{\psi} & & \searrow^{\mathcal{U}_\varepsilon \varphi} & \\ Y & \xrightarrow{[\![\bullet,0]\!]} & \mathcal{U}_\varepsilon Y & \xrightarrow{[\![\bullet,0]\!]} & \mathcal{U}_{2\varepsilon} Y \end{array} .$$

φ and ψ are called ε-isomorphisms if and only if both of the above diagrams commute.

Definition 53 (Interleaving distance)**.**

$$d_{\mathrm{I}}(X, Y) := \inf\left\{ \varepsilon \geq 0 \;\middle|\; \textit{there exist } \varepsilon\textit{-isomorphisms } \begin{pmatrix} \varphi\colon X \to \mathcal{U}_\varepsilon Y \\ \psi\colon Y \to \mathcal{U}_\varepsilon X \end{pmatrix} \right\}.$$

Later, we restrict the domain of the metric d_{I} to a particular subcategory of partially ordered trees so that we can show that the smoothing functor behaves much better there.

7 Stability of Reeb posets

In this section, we prove the stability theorem of Reeb posets.

7.1 Reeb posets

First, we prepare a category whose objects represent input data of the Reeb ordering method. The Reeb ordering method uses 0-th persistent homologies of sublevel and superlevel filtrations. Hence, naively speaking, we could have two options for input data types:

- a graph (or 1-simplicial complex) with real-values assigned at vertices, or
- a topological space with a real-valued continuous function from it.

We can construct sublevel and superlevel filtrations from “the adjacency structures” of either data type. However, the topological structures other than “the adjacency structures” are less critical because the Reeb ordering method depends only on the order structures obtained from the 0-dimensional persistent homologies. Thus, from the viewpoint of data analysis, we need an algorithm that computes the filtrations and reflects “the adjacencies” well. On the one hand, only the adjacency structures of input data are essential. On the other hand, the topological structures of output data are also important because they have something to do with the ε-smoothed CC-pospaces. Primarily, there is a situation where we need to deal with the smoothed CC-pospaces as input data again. Hence, we also consider the topological structure of input data just for this technical reason.

Definition 54 (Category of relational spaces)**.** *Let $(X, \mathcal{O}_X)$ be a topological space and R_X a homogeneous symmetric binary relation on X. We say $(X, \mathcal{O}_X, R_X)$ is a relational space. We define the category of relational spaces as follows.*

$$\operatorname{Ob}\mathbf{RelTop} = \{(X, \mathcal{O}_X, R_X) \mid X \text{ is an } R\text{-connected relational space}\},$$
$$\operatorname{Mor}\mathbf{RelTop} = \{\text{continuous relation-preserving maps}\}.$$

Here, we say $\varphi\colon (X, \mathcal{O}_X, R_X) \to (Y, \mathcal{O}_Y, R_Y)$ is relation-preserving if and only if, for all $x, x' \in X$, $x\ R_X\ x'$ implies $\varphi(x)\ R_Y\ \varphi(x')$.

For example, consider a simple connected undirected graph $G = (V, E)$ equipped with the discrete topology $\mathbf{2}^V$ on the set V of vertices. Note that we may identify the set E of simple undirected edges as a subset of V^2; namely, E is a symmetric binary relation of the adjacency on G. Thus, $(V, \mathcal{O}_V, E)$ is a relational space. We can regard the category of simple undirected graphs as a subcategory of **RelTop**.

Consider the following forgetful functor,

$$\mathcal{F}\colon \mathbf{CCPoHaus} \ni (X, \mathcal{O}_X, \leq) \longmapsto \big(X, \mathcal{O}_X, \lesseqgtr\big) \in \mathbf{RelTop}.$$

Here, $\lesseqgtr$ denotes the comparability relation $(\leq) \cup (\geq)$. When dealing with input data, we use $\mathcal{F}\mathbf{R} = (\mathbf{R}, \mathcal{O}_\mathbf{R}, \mathbf{R}^2)$ as a codomain. Note that $\mathbf{R}$ is totally ordered; thus, the resulting relation is entire. Hence, we can regard any real-valued continuous function as a morphism $X \to \mathcal{F}\mathbf{R}$ in **RelTop**.

Definition 55 (Reeb poset functor)**.** *Define $\mathcal{R}$ as $\mathcal{R}X \coloneqq (X, \mathcal{O}, \lesssim_\pm, f)/{\sim_\pm}$ for $(X, \mathcal{O}, R, f) \in \mathbf{RelTop}_{/\mathcal{F}\mathbf{R}}$. Here, $\lesssim_\pm$ denotes the Reeb quasi-order of f on the graph (X, R).*

Proposition 56. *$\mathcal{R}$ is a functor $\mathcal{R}\colon \mathbf{RelTop}_{/\mathcal{F}\mathbf{R}} \longrightarrow \mathbf{QoTop}_{/\mathbf{R}}$.*

Proof. By definition, $\mathcal{R}X$ is a poset endowed with a quotient topology. Hence, we have $\mathcal{R}X \in \mathbf{QoTop}_{/\mathbf{R}}$. (However, $\mathcal{R}X$ need not be a CC-pospace in general.) We check the functoriality of $\mathcal{R}$. Take $\varphi\colon (X, \mathcal{O}_X, R_X, f_X) \to (Y, \mathcal{O}_Y, R_Y, f_Y)$ in $\mathbf{RelTop}_{/\mathcal{F}\mathbf{R}}$ arbitrarily. φ preserves the height and relation. Hence, φ obviously preserves the Reeb quasi-order by its definition. Therefore, $q_Y \circ \varphi \circ q_X^{-1}$ is a map preserving the Reeb ordering, where $q_X\colon X \to \mathcal{R}X$ and $q_Y\colon Y \to \mathcal{R}Y$ denote the quotient maps, respectively. We thus obtain a height- and order-preserving continuous map $\mathcal{R}\varphi \coloneqq q_Y \circ \varphi \circ q_X^{-1}\colon \mathcal{R}X \to \mathcal{R}Y$. □

We call $\mathcal{R}$ the Reeb poset functor.

7.2 Stability theorem

Before proving the stability of the Reeb ordering method, we show the following essential properties which $\mathcal{R}$ enjoys.

Lemma 57 (Invariance of trees by $\mathcal{R}$)**.** *Let $(X, \mathcal{O}_X, \leq, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$. Assume that $(X, \leq_X)$ is a partially ordered tree, $\mathbf{MC}(X)$ branch complete, and f a strictly order-preserving height function. Then, $\mathcal{RF}X$ is a CC-pospace. Furthermore, we have the following isomorphism,*

$$X \cong \mathcal{RF}X \ \text{ in } \mathbf{CCPoHaus}_{/\mathbf{R}}.$$

Proof. Let $X \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ and assume that X is a partially ordered tree. Let $q\colon X \twoheadrightarrow \mathcal{RF}X$ be the quotient map. It is a continuous surjective order-preserving map. We check that q is injective.

Take $x, y \in X$ such that $q(x) = q(y)$ arbitrarily. It necessarily follows $f(x) = f(y) =: a$. By definition of Reeb ordering, we have $x \lesssim_+ y \lesssim_- x$, where $\lesssim_+$ and $\lesssim_-$ denote the quasi-orders on X induced by the 0-dimensional persistent homologies of superlevel and sublevel filtrations respectively. In other words, there are some paths $p^+ = \left(x_i^+\right)_{i=0}^n$ and $p^- = \left(x_i^-\right)_{i=0}^m$ from x to y on the graph $\mathcal{F}X$ such that

$$\begin{aligned} x_i^+ &\in f^{-1}[\uparrow a] \qquad (i = 0, 1, \dots, n), \\ x_i^- &\in f^{-1}[\downarrow a] \qquad (i = 0, 1, \dots, m). \end{aligned}$$

For all i, as $x_i^{\pm}$ and $x_{i+1}^{\pm}$ are comparable in X, there (uniquely) exists a locally maximal chain between them. Hence, we can identify p^+ and p^- as chain paths from x to y in $\mathbf{CPath}(X)$. As $\pi_1(X)$ is trivial, a path in normal form from x to y must be uniquely determined. Thus, applying strong normalisation to both p^+ and p^-, there is a unique chain path σ such that

$$p^+ \overset{\omega}{\Longrightarrow} \sigma \qquad \text{and} \qquad p^- \overset{\omega}{\Longrightarrow} \sigma.$$

Considering any explicit reduction sequences from p^+ and p^- to their normal form σ, we get that $f[\sigma] \subseteq f[p^+]$ and $f[\sigma] \subseteq f[p^-]$. Here, $f[\bullet]$ denotes the range of f of a chain path, which is the union of the ranges of f of all the chains forming the path. Hence, we obtain

$$f[\sigma] \subseteq f[p^+] \cap f[p^-] \subseteq \uparrow a \cap \downarrow a = \{a\}.$$

Thus, σ routes only on the level set $f^{-1}[a]$. By assumption on f and minimumness of σ, this is possible exactly when σ coincides with the empty path $\varnothing$. Therefore, as x and y are connected by σ, we necessarily have $x = y$ and conclude that q is injective.

Finally, as the quotient map q is a bijection, $\mathcal{RF}X$ is homeomorphic to the CC-pospace X. Therefore, $\mathcal{RF}X$ must be a CC-pospace, from which we conclude $q\colon X \xrightarrow{\sim} \mathcal{RF}X$ is an isomorphism in $\mathbf{CCPoHaus}_{/\mathbf{R}}$. □

As we have seen in the above lemma, the three properties of X, being a partially ordered tree, $\mathbf{MC}(X)$ branch complete, and f strictly order-preserving, play an important role. Remind that chain finiteness, chain final topology, and local compactness of chains are essential properties as well supporting Lemmas 29 and Corollary 45. Hence, we introduce the category consisting of CC-pospaces with such properties.

Definition 58 (**R**-tree)**.** *Let $(X, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$. We say (X, f) is an* **R***-tree if and only if the following are satisfied:*

- *X is a chain finite partially ordered tree,*
- *X is endowed with a chain final topology,*
- *all locally maximal chains in X are compact, and*
- *f is strictly order-preserving.*

Let **R-Tree** *denote the subcategory of $\mathbf{CCPoHaus}_{/\mathbf{R}}$ that consists of all* **R***-trees.*

Remark 59. *Let $(X, f) \in \mathbf{CCPoHaus}_{/\mathbf{R}}$ and $\varepsilon \geq 0$. In general, the morphism $[\![\bullet, 0]\!]\colon X \to \mathcal{U}_\varepsilon X$ does not induce a functor $\mathbf{MC}(X) \to \mathbf{MC}(\mathcal{U}_\varepsilon X)$ canonically. Let us consider the following poset X,*

$$X = \left(\begin{array}{ccccccccccc} 1_a & \longrightarrow & 2_a & \longrightarrow & 3_a & \longrightarrow & 4_a & \longrightarrow & 5_a & \longrightarrow & 6_a \\ & \searrow & & & & & & & & \nearrow & \\ & & 2_b & & & & & & 5_b & & \\ & \nearrow & & & & & & & & \searrow & \\ 1_c & & & & \longrightarrow & & & & & & 6_c \end{array} \right).$$

Set the height $f(i_t) = i$, removing the label $t \in \{a, b, c\}$ from a labelled number $i_t \in X$. Let X be equipped with the discrete topology. Then, the 1-smoothed CC-pospace of (X, f) can be expressed as follows.

$$\mathcal{U}_1(X, f) = \operatorname{colim} \left(\begin{array}{ccccccc} & & [1,2] & \hookrightarrow & [1,3]_b & \hookleftarrow & [1,2] & & \\ & \swarrow & & & & & & \searrow & \\ [0,7]_a & & & & & & & & [0,2]_c \cup [5,7]_c \\ & \nwarrow & & & & & & \nearrow & \\ & & [5,6] & \hookrightarrow & [4,6]_b & \hookleftarrow & [5,6] & & \end{array} \right).$$

Here, each labelled object in the above diagram is a copy of a chain in **R** *with different labels. Note that the unlabelled intervals are no longer distinguished in $\mathcal{U}_1 X$. We originally had two locally maximal chains in X from 1_t to 6_t (for $t = a, c$). However, there is only one locally maximal chain from 1 to 6 in $\mathcal{U}_1(X, f)$, which is $[1, 6]$.*

Thus, a smoothing functor does not generally preserve the number of locally maximal chains. Using this fact, we can construct a worse example $X' = \operatorname{colim}(X \leftarrow \{1_c \to 6_c\} \to X)$, in which we attach two copies of X along the c-labelled elements. There are three locally maximal chains from a height 1 to a height 6 in X', but only two in $\mathcal{U}_1 X'$. Therefore, we can not construct a canonical functor $\mathbf{MC}(X') \to \mathbf{MC}(\mathcal{U}_1 X')$ induced by the morphism $[\![\bullet, 0]\!]\colon X' \to \mathcal{U}_1 X'$.

Lemma 60 ($\mathcal{U}_\varepsilon$ preserves **R-Tree**)**.** *Let* $(X, f) \in \mathbf{R}\text{-}\mathbf{Tree}$ *and* $\varepsilon \geq 0$. *Then, the morphism* $[\![\bullet, 0]\!]\colon X \to \mathcal{U}_\varepsilon X$ *induces a functor* $\mathbf{MC}(X) \to \mathbf{MC}(\mathcal{U}_\varepsilon X)$. *Especially,* $\mathcal{U}_\varepsilon X \in \mathbf{R}\text{-}\mathbf{Tree}$.

Proof. Let $(X, f) \in \mathbf{R}\text{-}\mathbf{Tree}$. Thanks to Lemmas 40 and 51, we can write $X = \operatorname{colim} \mathcal{D}$ for some finite diagram $\mathcal{D}\colon \mathbf{I}(X) \to \mathbf{CCPoHaus}_{/\mathbf{R}}$ in which objects are chains in X. Furthermore, $\mathcal{D}$ includes all the maximal chains in X. Applying Lemma 50, we obtain $\mathcal{U}_\varepsilon X \cong \operatorname{colim} \mathcal{U}_\varepsilon \mathcal{D}$. Take $C\colon x \to y$ in $\mathbf{MC}(X)$ arbitrarily. There is some i such that $\mathcal{D}i$ is maximal in X and includes C. Let C' denote the image of $[\![\bullet, 0]\!]$ of C. Suppose C' is not locally maximal for contradiction. Then, there is some locally maximal chain $\bar{C}'$ in $\mathcal{U}_\varepsilon X$ including C'. We have $\bar{C}' \setminus C' \neq \emptyset$ by assumption. Take some $[\![z, t]\!] \in \bar{C}' \setminus C'$. There is some subdiagram $\mathcal{D}|_I$ of $\mathcal{D}$ such that $\operatorname{colim} \mathcal{U}_\varepsilon \mathcal{D}|_I$ includes $\bar{C}'$ and I is an index category with $i \in I \subseteq \mathbf{I}(X)$. As $z \notin \mathcal{D}i$, there is some $j \in I$ with $z \in \mathcal{D}j$. As $[\![x, 0]\!] \leq [\![z, t]\!] \leq [\![y, 0]\!]$ and $x, y \in \mathcal{D}i$, there exists a path $(i_0, \ldots, i_n)$ in the index category I from i via j to i. However, since such a path extends to a non-trivial cycle in $\mathbf{CPath}(X)$, it contradicts the property $\pi_1(X) = 0$. Therefore, C' must be locally maximal, and we obtain $C'\colon [\![x, 0]\!] \to [\![y, 0]\!]$ in $\mathbf{MC}(\mathcal{U}_\varepsilon X)$. The functoriality of $[\![\bullet, 0]\!]$ is trivial; hence, we conclude that $[\![\bullet, 0]\!]$ induces a functor from $\mathbf{MC}(X)$ to $\mathbf{MC}(\mathcal{U}_\varepsilon X)$. Finally, the smoothed pospace $\mathcal{U}_\varepsilon X$ is an $\mathbf{R}$-tree, thanks to Theorem 52. □

Let us revisit the interleaving distance in **R-Tree**. Let $(X, f), (Y, g) \in \mathbf{R}\text{-}\mathbf{Tree}$. Recall that the interleaving distance is defined as follows,

$$d_{\mathrm{I}}(X, Y) \coloneqq \inf\left\{ \varepsilon \geq 0 \;\middle|\; \text{there exist } \varepsilon\text{-isomorphisms } \begin{pmatrix} \varphi\colon X \to \mathcal{U}_\varepsilon Y \\ \psi\colon Y \to \mathcal{U}_\varepsilon X \end{pmatrix} \right\}.$$

Note that, thanks to Lemma 60, both $\mathcal{U}_\varepsilon X$ and $\mathcal{U}_\varepsilon Y$ are **R**-trees. Hereafter, we restrict the domain of the interleaving distance to **R-Tree**. We can show that d_{I} is indeed an extended pseudometric.

Proposition 61. $d_{\mathrm{I}}\colon \mathbf{R}\text{-}\mathbf{Tree} \times \mathbf{R}\text{-}\mathbf{Tree} \to \mathbf{R}_{\geq 0} \cup \{\infty\}$ *is an extended pseudometric; namely, it satisfies* $d_{\mathrm{I}}(X, X) = 0$, $d_{\mathrm{I}}(X, Y) = d_{\mathrm{I}}(Y, X)$, *and* $d_{\mathrm{I}}(X, Z) \leq d_{\mathrm{I}}(X, Y) + d_{\mathrm{I}}(Y, Z)$ *for all* $X, Y, Z \in \mathbf{R}\text{-}\mathbf{Tree}$. *Furthermore,* d_{I} *satisfies the following.*

- *If there is some isomorphism between* $X, Y \in \mathbf{R}\text{-}\mathbf{Tree}$, *we have* $d_{\mathrm{I}}(X, Y) = 0$.
- *There exist some* X *and* Y *with* $X \ncong Y$, *but* $d_{\mathrm{I}}(X, Y) = 0$.
- *There exist some* X *and* Y *with* $d_{\mathrm{I}}(X, Y) = \infty$.

Proof. We have the symmetry law by definition and the triangle inequality by the composition of ε-isomorphisms.

If there is an isomorphism $\varphi\colon X \to Y$ in **R-Tree**, φ is a 0-isomorphism, from which $d_{\mathrm{I}}(X, Y) = 0$ follows. Note that this includes the case $X = Y$ and $\varphi = \mathrm{id}_X$.

Set $X = [0, 1)$ and $Y = (0, 1]$ both in $\mathbf{R}$, clearly with $X \ncong Y$. X and Y are $\mathbf{R}$-trees endowed with the height themselves. For every $\varepsilon > 0$, there are unique height-preserving maps $X \to \mathcal{U}_\varepsilon Y$ and $Y \to \mathcal{U}_\varepsilon X$ which form ε-isomorphisms. Note $\mathcal{U}_\varepsilon X \cong [-\varepsilon, 1 + \varepsilon) \supset Y$. Hence, we have $d_{\mathrm{I}}(X, Y) \leq \inf_{\varepsilon > 0} \varepsilon = 0$.

Set $X = [0, \infty)$ and $Y = (-\infty, 0]$ both in $\mathbf{R}$. X and Y are $\mathbf{R}$-trees endowed with the height themselves. For every $\varepsilon > 0$, there are no height-preserving maps from X to $\mathcal{U}_\varepsilon Y$ nor Y to $\mathcal{U}_\varepsilon X$. Note $\mathcal{U}_\varepsilon X \cong [-\varepsilon, \infty)$. We thus have $d_{\mathrm{I}}(X, Y) = \infty$. □

Theorem 62 (Stability of Reeb posets)**.** *Let $X \in$ **RelTop** and $(X,f),(X,g) \in$ $\mathbf{RelTop}_{/\mathcal{F}\mathbf{R}}$. Assume both $\mathcal{R}(X,f)$ and $\mathcal{R}(X,g)$ are* **R***-trees. Then, we have the following inequality:*

$$d_{\mathrm{I}}(\mathcal{R}(X,f),\mathcal{R}(X,g)) \leq \|f-g\|_{\infty}.$$

Proof. Set $(T,f_T) \coloneqq \mathcal{R}(X,f)$, $(S,g_S) \coloneqq \mathcal{R}(X,g)$, and $\varepsilon \coloneqq \|f-g\|_{\infty}$. Let $q_T\colon X \to T$ and $q_S\colon X \to S$ denote the quotient maps, respectively. We construct a pair of ε-isomorphisms between T and S. Consider the following map,

$$\sigma\colon X \ni x \longmapsto [\![q_S(x), f(x)-g(x)]\!]_{\mathcal{U}_\varepsilon S} \in \mathcal{F}\mathcal{U}_\varepsilon S.$$

Then, $\sigma\colon (X,f) \to \mathcal{F}\mathcal{U}_\varepsilon(S,g_S)$ preserves the height and relation. Applying the Reeb poset functor (Definition 55 and Proposition 56), we get $\mathcal{R}\sigma\colon T \to \mathcal{R}\mathcal{F}\mathcal{U}_\varepsilon S$. As $\mathcal{U}_\varepsilon S$ is an $\mathbf{R}$-tree by assumption and Lemma 60, there is an isomorphism $r_S\colon \mathcal{R}\mathcal{F}\mathcal{U}_\varepsilon S \xrightarrow{\sim} \mathcal{U}_\varepsilon S$ given by Lemma 57. Combining these, we obtain $\varphi \coloneqq r_S \circ \mathcal{R}\sigma\colon T \to \mathcal{U}_\varepsilon S$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$. Similarly, we define $\psi \coloneqq r_T \circ \mathcal{R}\tau\colon S \to \mathcal{U}_\varepsilon T$ in $\mathbf{CCPoHaus}_{/\mathbf{R}}$, where $\tau\colon x \longmapsto [\![q_T(x), g(x)-f(x)]\!]_{\mathcal{U}_\varepsilon T}$ and $r_T\colon \mathcal{R}\mathcal{F}\mathcal{U}_\varepsilon T \xrightarrow{\sim} \mathcal{U}_\varepsilon T$. Then, the pair (φ,ψ) satisfies $\mathcal{U}_\varepsilon\psi\circ\varphi = [\![\bullet,0]\!]_{\mathcal{U}_{2\varepsilon}T}$ and $\mathcal{U}_\varepsilon\varphi\circ\psi = [\![\bullet,0]\!]_{\mathcal{U}_{2\varepsilon}S}$; namely, they are ε-isomorphisms. Hence, we conclude $d_{\mathrm{I}}(T,S) \leq \varepsilon$. □

Proposition 63. *Let $X \coloneqq \{a_0,\dots,a_n\}$ and $Y \coloneqq [a_0,a_n]$ be subspaces of* **R** *regarded as CC-pospaces, where $a_0,\dots,a_n \in \mathbf{R}$ are in the ascending order $a_0 \leq \cdots \leq a_n$. Then, we have the following,*

$$d_{\mathrm{I}}(X,Y) = \frac{1}{2}\max_{0\leq i<n}(a_{i+1}-a_i). \tag{7}$$

Proof. Let $\varepsilon \geq 0$ and δ the right-hand side of (7). We have inclusions $X \hookrightarrow Y \hookrightarrow \mathcal{U}_\varepsilon Y$. These are the unique height-preserving morphisms from X via Y to $\mathcal{U}_\varepsilon Y$ as all involved sets are totally ordered. Suppose there exists a height-preserving morphism $\psi\colon Y \to \mathcal{U}_\varepsilon X$. Then, since $[a_0,a_n] \cong \psi[Y] \subseteq \mathcal{U}_\varepsilon X \cong \cup_i(a_i+[-\varepsilon,\varepsilon])$, it follows that $\varepsilon \geq \delta$. Conversely, for every $\varepsilon \geq \delta$, we can construct such a height-preserving map ψ, which is the unique morphism from Y to $\mathcal{U}_\varepsilon X$. This ψ is an ε-isomorphism by definition. Therefore, taking the infimum over all feasible values of ε, we conclude $d_{\mathrm{I}}(X,Y) = \delta$. □

Note that in the above proof, Y serves as a unique linear interpolation of X. When comparing a pair of CC-pospaces, mainly when one is a linear interpolation of the other, one might anticipate a similar value to bridge the gap between discrete and continuous data. However, applying the same proof to general scenarios where **R**-trees are not totally ordered is non-trivial. Generally, identifying or enumerating all structure-preserving morphisms between two pospaces poses combinatorial challenges. Despite these hurdles, half the size of the data gap is always noteworthy.

Theorem 64 (Comparison with continuous version)**.** *Let $(X,f),(Y,g) \in \mathbf{Top}_{/\mathbf{R}}$ be simply connected spaces endowed with height functions. Assume there are some triangulations of X and Y where f and g become piecewise linear. Let R_{X_0} and R_{Y_0} denote* *the adjacency relations over the sets X_0 and Y_0 of vertices on these triangulations. Then, we have the following inequality,*

$$d_{\mathrm{R}}(\tilde{\mathcal{R}}(X,f),\tilde{\mathcal{R}}(Y,g)) \le d_{\mathrm{I}}(\mathcal{R}(X_0,R_{X_0},f|_{X_0}),\mathcal{R}(Y_0,R_{Y_0},g|_{Y_0})). \tag{8}$$

The left-hand side is the Reeb distance between the two Reeb graphs.

The key is that ε-isomorphisms in $\mathbf{CCPoHaus}_{/\mathbf{R}}$ require matching between semi-discrete data points, whereas those in $\mathbf{Top}_{/\mathbf{R}}$ do not possess such delicate information. Hence, the morphisms in $\mathbf{Top}_{/\mathbf{R}}$ do not help us construct the matching between semi-discrete $\mathbf{R}$-trees sufficiently.

Proof. Claim: we have the following equality,

$$d_{\mathrm{R}}(\tilde{\mathcal{R}}(X,f),\tilde{\mathcal{R}}(Y,f)) = d_{\mathrm{I}}(\mathcal{R}(X,R_X,f),\mathcal{R}(Y,R_Y,g)),$$

where R_X (resp. R_Y) denote the neighbourhood relation over X (resp. Y) induced by the closed stars on the triangulation of X (resp. Y). To see this, recall that the Reeb graph $\tilde{\mathcal{R}}X$ coincides with the Reeb poset $\mathcal{R}X$ as discussed in Section 2. More precisely, they are isomorphic in $\mathbf{CCPoHaus}_{/\mathbf{R}}$ by equipping a climbing order with the Reeb graph. Also, we need to fill in the gap of difference between the smoothing functors $\tilde{\mathcal{U}}_\varepsilon$ and $\mathcal{U}_\varepsilon$. It is actually almost straightforward by definition; $\tilde{\mathcal{U}}_\varepsilon$ is defined by the Reeb graph of a product space, and $\mathcal{U}_\varepsilon$ by the pospace reflection of a product space endowed with a climbing order. We can quickly check that the same equivalence relation over the product space is used to take the quotient to obtain the smoothed spaces. Thus, we can transform any ε-isomorphism between the Reeb graphs $\tilde{\mathcal{R}}X$ and $\tilde{\mathcal{R}}Y$ to that between the corresponding Reeb posets and vice versa. This correspondence proves the above equality.

Second, we compare $\mathcal{R}(X,R_X,f)$ and $\mathcal{R}(X_0,R_{X_0},f|_{X_0})$. The inclusion $X_0 \hookrightarrow X$ is relation-preserving; in other words, we have $R_{X_0} \subset R_X$ $(\subset X^2)$. Thus, it induces $\mathcal{R}X_0 \hookrightarrow \mathcal{R}X$. Furthermore, we can regard $\mathcal{R}X$ as a linear interpolation of $\mathcal{R}X_0$. To obtain this, it is indeed feasible to make "discrete" intervals $[a,b]$ in $\mathcal{R}X_0$ correspond to "continuous" real-valued intervals $[\mathcal{R}f(a),\mathcal{R}f(b)]$ $(\subset \mathbf{R})$, transforming them while suitably patching them together. (Note that this simple construction is possible only when (X_0,X_1) is a part of the triangulation of X that is simply connected. For the multiply connected case, the construction must be done more carefully as intervals in a poset may not be totally ordered.) Similarly, for Y_0 and Y, we have the inclusion $\mathcal{R}Y_0 \hookrightarrow \mathcal{R}Y$ to the linear interpolation.

Third, we construct ε-isomorphisms. The desired inequality is trivial if $d_{\mathrm{I}}(\mathcal{R}X_0,\mathcal{R}Y_0)=\infty$. Hence, we may assume $d_{\mathrm{I}}(\mathcal{R}X_0,\mathcal{R}Y_0)<\infty$ without loss of generality. For any $\varepsilon \ge d_{\mathrm{I}}(\mathcal{R}X_0,\mathcal{R}Y_0)$, let $\varphi_0\colon \mathcal{R}X_0 \to \mathcal{U}_\varepsilon\mathcal{R}Y_0$ and $\psi_0\colon \mathcal{R}Y_0 \to \mathcal{U}_\varepsilon\mathcal{R}X_0$ be ε-isomorphisms. Extending these along the linear interpolation, we obtain $\varphi\colon \mathcal{R}X \to \mathcal{U}_\varepsilon\mathcal{R}Y$ and $\psi\colon \mathcal{R}Y \to \mathcal{U}_\varepsilon\mathcal{R}X$. (In other words, we apply the linear interpolation functor to φ_0 and ψ_0.) φ and ψ are ε-isomorphisms between $\mathcal{R}X$ and $\mathcal{R}Y$, which means $d_{\mathrm{I}}(\mathcal{R}X,\mathcal{R}Y) \le \varepsilon$ by definition. Therefore, by taking the infimum, we conclude $d_{\mathrm{I}}(\mathcal{R}X,\mathcal{R}Y) \le d_{\mathrm{I}}(\mathcal{R}X_0,\mathcal{R}Y_0)$. □

Observing Proposition 63 and Theorem 64, one might anticipate that the difference between the two metrics takes some value close to half the size of the data gap. However, it is not straightforward to show such an explicit estimate due to the combinatorial challenges. Again, the main difficulty lies in the inability to enumerate all the structure-preserving morphisms. One solution is to compare only locally, without comparing globally. When the matching between the two pospaces can be given, locally, there is an equality (7), which approximately measures the difference. Many such estimates still have to be shown, but the author leaves them as future work in this paper.

8 Concluding remarks

In summary, this paper has introduced novel mathematical tools and frameworks to address the stability of the Reeb ordering method, particularly in the context of interleaving distance. We have extended the theory of ordered spaces and posets to offer a more robust understanding of the Reeb ordering method's semi-discrete nature. Our findings contribute to the theoretical landscape and have practical implications for topological data analysis. Future work may explore further generalisations and applications of our methods.

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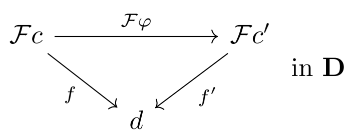
Appendix A Over category and under category

Let $\mathbf{C}, \mathbf{D}$ be categories and $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ a functor. Let $d \in \mathbf{D}$ be an object. We define the category of objects $\mathcal{F}$-over d as follows:

$$\begin{aligned} \mathrm{Ob}(\mathcal{F}_{/d}) &= \{(c, f) \mid f\colon \mathcal{F}c \to d \text{ in } \mathbf{D}\}, \\ \mathrm{Mor}(\mathcal{F}_{/d}) &= \{\varphi\colon (c, f) \to (c', f') \mid \varphi\colon c \to c' \text{ in } \mathbf{C} \text{ with } f = f' \circ \mathcal{F}\varphi\}. \end{aligned}$$

The term "($\mathcal{F}$-)over" came from a diagram where a morphism is drawn horizontally, and objects $\mathcal{F}$-over d are located above d as follows:



Dually, we define the category of objects $\mathcal{F}$-under d as follows:

$$\begin{aligned}\mathrm{Ob}({}^{d/}\mathcal{F}) &= \{(c, f) \mid f\colon d \to \mathcal{F}c \text{ in } \mathbf{D}\},\\ \mathrm{Mor}({}^{d/}\mathcal{F}) &= \{\varphi\colon (c, f) \to (c', f') \mid \varphi\colon c \to c' \text{ in } \mathbf{C} \text{ with } f' = \mathcal{F}\varphi \circ f\}.\end{aligned}$$

If $\mathbf{C} = \mathbf{D}$ and $\mathcal{F} = \mathrm{Id}$, we write $\mathbf{D}_{/d}$ instead of $\mathcal{F}_{/d}$. The category $\mathbf{D}_{/d}$ is also known as an over category or a slice category. An object of $\mathbf{D}_{/d}$ is a pair (x, f) of an object $x \in \mathbf{D}$ and a morphism $f : x \to d$ in $\mathbf{D}$. In our theory, we frequently deal with objects over $\mathbf{R}$ to deal with a pair of a space endowed with a scalar data representing height such as a contour plot ($\mathbf{Top}_{/\mathbf{R}}$) and a Reeb graph ($\mathbf{CCPoHaus}_{/\mathbf{R}}$).

The concept of these categories is generalised to a so-called comma category. The following fact is well-known in category theory regarding the $\mathcal{F}$-under category. For instance, see Proposition 2.5.2 in Kashiwara and Schapira (2010) for details.

Fact 65. *Let $\mathcal{F}\colon \mathbf{C} \to \mathbf{D}$ be a functor. Then, the following are equivalent:*

- *$\mathcal{F}$ is final; namely, for all $x \in \mathbf{D}$, the category ${}^{x/}\mathcal{F}$ of objects $\mathcal{F}$-under x is connected.*
- *For any category $\mathbf{E}$ and diagram $\mathcal{D}\colon \mathbf{D} \to \mathbf{E}$, the natural morphism $\mathrm{colim}(\mathcal{D}\mathcal{F}) \to \mathrm{colim}\,\mathcal{D}$ is an isomorphism.*

Appendix B Notations

Listed in Table B1 are the notations in this article.

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Table B1 Notation

General notations	
$\mathbf{N}$	the set of all non-negative integers
$\mathbf{Z}$	the set of all integers
$\mathbf{R}$	the set of all real numbers
$\bullet$	an argument placeholder of a function used in the form $f(\bullet)$
$\hookrightarrow$	an injection (or especially, an inclusion)
$\twoheadrightarrow$	a surjection (or especially, a quotient map)
$\xrightarrow{\sim}$	a bijection (or especially, an isormorphism)
$a, b, \ldots, x, y, \ldots$	elements or objects
$f, g, h, \ldots$	maps or morphisms, typically denoting height functions
$\varphi, \psi, \ldots$	morphisms, typically denoting structure-preserving morphisms
$X, Y, Z, \ldots$	sets or spaces, typically denoting ordered spaces
C, C'	(locally maximal) chains in a poset
$f[A]$	the image of f of A
$f^{-1}[A]$	the inverse image of f of A
$f\|_A$	the restriction of f to A
$\mathbf{C}, \mathbf{D}, \ldots$	categories
$\operatorname{Ob} \mathbf{C}$	the class of all objects in $\mathbf{C}$
$\operatorname{Mor} \mathbf{C}$	the class of all morphisms in $\mathbf{C}$
$\operatorname{Hom}_{\mathbf{C}}(a, b)$	the hom-set of all morphisms from a to b in $\mathbf{C}$
$\mathcal{F}, \mathcal{D}, \ldots$	functors or diagrams
$\operatorname{colim}_{\mathbf{C}} \mathcal{D}$	the colimit of $\mathcal{D}$ in $\mathbf{C}$
f^{op}, $\mathbf{C}^{\mathrm{op}}$	the opposite morphism of f and the opposite category of $\mathbf{C}$
$\mathbf{Free}(G)$	the free category generated by G
$\mathbf{C}[R^{-1}]$	the localisation of $\mathbf{C}$ along R
$\mathbf{C}_{/a}$	the over (slice) category of $\mathbf{C}$ over $a \in \mathbf{C}$
Specific notations	
$\lesssim, \leq$	a quasi-order (preorder) and a partial order
(R)	the graph of R, typically for the infix relational notation $\bullet\ R\ \bullet$
$\lesseqgtr$	a comparability relation; $(\lesseqgtr) = (\leq) \cup (\geq)$
$\uparrow, \downarrow$	an upset closure operator and a downset closure operator
$\lesssim_{f\uparrow}, \leq_{f\uparrow}$	f-climbing orders (Definition 10)
${}_f{\sqcup}_g$, $\sqcup$, ${}_f{\sqcap}_g$, $\sqcap$	initial branches and final branches (Definition 27)
R^*	the (reflexive) transitive closure of R
$\stackrel{\tau}{\Longrightarrow}$	a τ-reduction for $\tau = \alpha, \beta, \gamma, \delta, \omega$ (Definitions 33 and 38)
$\mathbf{PoSet}$	the category of posets (partially ordered sets)
$\mathbf{Top}$	the category of topological spaces
$\mathbf{QoTop}$	the category of quasi-ordered spaces
$\mathbf{CCPoHaus}$	the category of CC-pospaces (Section 3)
$\mathbf{RelTop}$	the category of relational spaces (Definition 54)
$\mathbf{MC}(X)$	the category of locally maximal chains in X (Definition 14)
$\mathbf{CPath}(X)$	the category of chain paths in X (Definition 16)
$\mathbf{LPath}(X)$	the category of localised chain paths in X (Definition 16)
$\pi_1(X; *)$	the fundamental group of X (Definition 18)
$\mathbf{Ch}(X)$	the category of chains in X (Section 6.2.1)
$\mathbf{I}(X)$	the index category ($\hookrightarrow \mathbf{Ch}(X)$), typically finite (Section 6.2.1)
$\mathbf{R}$-$\mathbf{Tree}$	the category of $\mathbf{R}$-trees (Definition 58)
$\mathcal{P}\colon \mathbf{QoTop} \to \mathbf{CCPoHaus}$	the CC-pospace reflector (Definition 8 and Proposition 9)
$\tilde{\mathcal{U}}_\varepsilon, \mathcal{U}_\varepsilon\colon \mathbf{C} \to \mathbf{C}$	ε-smoothing functors (Section 6 and Definition 46)
$\mathcal{F}\colon \mathbf{CCPoHaus} \to \mathbf{RelTop}$	the forgetful functor (Section 7)
$\mathcal{R}\colon \mathbf{RelTop}_{/\mathcal{F}\mathbf{R}} \longrightarrow \mathbf{QoTop}_{/\mathbf{R}}$	the Reeb poset functor (Definition 55)
$[\![\bullet, \bullet]\!]\colon X \times [-\varepsilon, \varepsilon] \to \mathcal{U}_\varepsilon X$	the quotient map to the ε-smoothed pospace