Delay Switching across the Stability Boundary ⋆

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Abstract: We propose a simple delay differential equation with a delay switching. In this model, the delay is a time-dependent variable taking two values across the stability boundary. With both stochastic and regular periodic switching of the delay, there are cases where the region of asymptotic stability is enhanced. We also show that this is in contrast to the analogous case of switching coefficient parameters in the equation. Also, the direction of switching across the stability boundary affects the stability.

Keywords: Delay, Feedback, Stochasticity, Asymptotic Stability, Stability Control, Switched Systems

1. INTRODUCTION

Time delays are known to induce rather intricate behaviors in simple dynamical systems. For example, even for first-order ordinary differential equations, the stability of the fixed point changes by increasing delay. As time delays are almost ubiquitous in feedback control systems or multi-body interacting systems, the delay differential equations have been actively employed in theoretical modelings in various fields(1; 2; 3; 4; 5; 6; 7; 8; 9). Mathematical aspects of delay differential equations have also been investigated. Due to the non-linearity introduced by the delay, however, there still are many unsolved problems analytically.

Also, when we introduce stochasticity additionally, the study of the dynamics becomes more challenging. Even though there has been a series of investigations on such delayed stochastic systems(10; 11; 12; 13; 14; 15), understandings of the interplay between delay and stochasticity are yet to come.

What we propose here is a slightly modified simple delay differential equation that gives rise to yet another curious behavior. In our model, the delay is a variable taking two values: one in the stability and the other in the unstable regions of a fixed point. We have considered both cases of stochastic and periodic (regular) switchings, and have observed that the region of the asymptotic stability of the fixed point is enhanced by this switching of delay. We also investigate that this is in contrast to the analogous case of switching coefficient parameters in the equation.

After this initial investigation(16; 17), it has come to the attention of the author that similar investigations have been done in the field of control engineering(18; 19; 20; 21; 22; 23). In these works, however, the switchings are done either with the delay or with the coefficient parameter separately. We conjecture that there is the optimal direction of crossing the boundary by switching both of them together. Preliminary simulation results that support this conjecture are presented.

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2. MODEL

The basic equation we start with is the following simple delay differential equation:

$$\frac{dX(t)}{dt} = aX(t-\tau) \tag{1}$$

where a is a real parameter τ is the delay. This equation is known as a special case of Hayes equation(1), which has been much investigated. It is known that non-zero delay induces oscillations and for $a < 0$ the asymptotic stability of the origin $X = 0$ is lost for the delay larger than the critical value

$$\tau_c = -\frac{\pi}{2a}. \tag{2}$$

The stability boundary is shown in Figure 1.

2.1 Alternate Switching

We now extend this model so that the delay is changed to a real variable $\hat{\tau}$ in such a way that it takes two values across the critical delay. Specifically, it is given by the following

$$\hat{\tau}(t) = \tau_c(1 + \mu\xi(t)). \tag{3}$$

where $\mu \in (0, 1)$ is a real parameter, and ξ is a variable taking $+1$ and -1 periodically.

In order to gain insight, we discretize the equation,

$$X(t+1) = X(t) + a(dt)X(t - N[\hat{\tau}(t)/dt]), \tag{4}$$

where dt is a time discretization parameter, and $N[s]$ is a function which returns the closest integer to s.

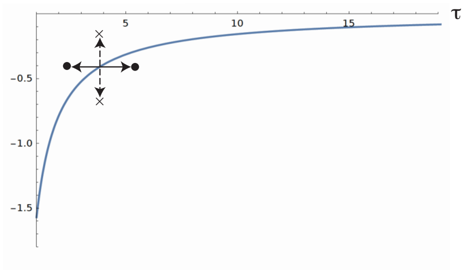

Fig. 1. The stability boundary of the origin of Equation (1). The area with the curve and the axes is the stable region. Also, the schematics of the delay switchings are shown. Delay switching is between the two dot points indicated by the solid horizontal arrows, while the switching for the coefficient a is indicated by the two cross points with the dashed vertical arrows.

In what follows, we investigate this discretized map (4). One should be careful that discretization could have different characteristics from the original continuous-time delay differential equation.

As a first step, we let ξ take $+1$ and -1 alternatively at every time step (i.e., period 2).

$$\hat{\tau}(t) = \begin{cases} \tau_c(1+\mu) > \tau_c \ (t \ even) \\ \tau_c(1-\mu) < \tau_c \ (t \ odd) \end{cases} \tag{5}$$

We set parameters $a = -1.5$ and $dt = 0.01$. Then, the critical delay for Eq. (1) is $\tau_c \approx 1.0472$. We now numerically simulate the discrete map (4), and obtain the behaviors as shown in Fig. 2.

We note that with the fixed delay $\tau = 1.05$ (i.e., $\mu = 0$), the origin is not stable and the dynamical trajectory diverges. This is expected as the value of the delay is larger than the critical value τ_c. As we increase μ so that the delay takes two values across the stability boundary with sufficient amplitude, the stability of the origin is obtained. Thus, even though the average of the values of the switching delays does not change, the change in the stability characteristics emerges by this alternating switching.

2.2 Stochastic Switching

We consider the case so that the delay is changed to a real stochastic process $\hat{\tau}(t)$ in such a way that it takes two values across the critical delay. As in the previous model, it is given by the following

$$\hat{\tau}(t) = \tau_c(1 + \mu\xi(t)). \tag{6}$$

where $\mu \in (0, 1)$ is a real parameter, and $\xi(t)$ is a stochastic process taking $+1$ and -1 with probabilities p and $1-p$ respectively. Thus,

$$\hat{\tau}(t) = \begin{cases} \tau_c(1+\mu) > \tau_c, \ (with \ the \ probability \ p) \\ \tau_c(1-\mu) < \tau_c, \ (with \ the \ probability \ 1-p) \end{cases} \tag{7}$$

So, as mentioned, the delay switches stochastically across the critical value of the delay as indicated by the solid horizontal arrows in Figure 1.

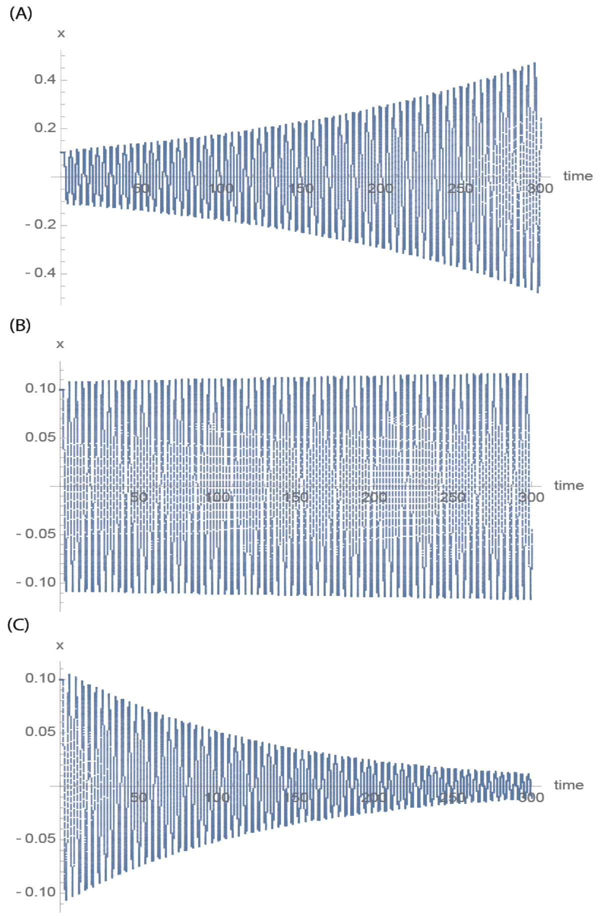

Fig. 2. A representative plot of the dynamical path of Eq. (4) with the alternating delay switching. The parameter values are $a = -1.5$ and $dt = 0.01$ The switching amplitude μ and corresponding delays are the following: (A)$\mu = 0, \tau = 1.05$, (B) $\mu = 0.08, \tau = (0.97, 1.13)$, (C) $\mu = 0.13, \tau = (0.92, 1.18)$.

We study this equation with stochastic delay switching numerically. The dynamical map (4) is now a stochastic delay switching map. The notable observation is that this switching enhances the stability of the origin: The origin can be asymptotically stable even when the average delay value, $\langle \hat{\tau} \rangle$, is larger than τ_c. We can view this stochastic switching with associated behaviors as yet another example showing the intricate interplay between delay and stochasticity.

2.3 Comparison against Coefficient Switching

Also, we compare these results with the case of the stochastic stability crossing by switching two values of a. Namely, we consider the following as a comparison,

$$\frac{dX(t)}{dt} = \hat{a}(t)X(t-\tau) \tag{8}$$

where $\hat{a}(t)$ is now the alternating or the stochastic variable taking two values across the stability boundary with a given delay as indicated by the dashed vertical arrows in Figure 1. This type of equation for the case of stochastic switchings can be considered as a special case of delay differential equations with a parametric noise(7). For our interest, we observe that the analogous stability enhancement does not occur in contrast to the case of the delay switching.

3. STABILITY CHART BY NUMERICAL SIMULATIONS

We have performed the numerical simulation of the stochastic map (4), and constructed a stability diagram as shown in Figure 2. The bottom figure is the log-log plot. As in Figure 1, The solid line is the stability boundary without stochastic switching. Here, the average delay value for the stochastic delay switching, $\langle\hat{\tau}\rangle_c$ are numerically estimated and indicated with the solid points on the plot. The enhancement of the stability region by stochastic delay switching is clearly observed as $\tau_c < \langle\hat{\tau}\rangle_c$.

Also, as we mentioned we have performed the stochastic stability crossing by switching two values of a with Equation (8). Representative results are also shown in Figure 2 with estimated average critical values $\langle\hat{a}\rangle_c$ indicated by the crosses. We see that the stability enhancement does not occur. In concrete, the values of the average critical values of $\hat{a}$ lie on the solid line so that $\langle\hat{a}\rangle_c \approx a_c$.

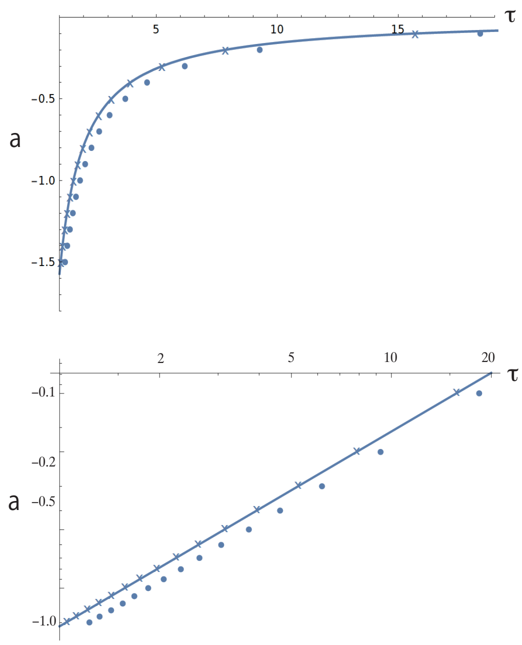

Fig. 3. A representative plot of the stability boundary of the origin of Equation (1) with the stochastic switching. The bottom figure is the log-log plot. Also, the results of stochastic switching of the coefficients a are indicated by the crosses that show no stability enhancement. (The parameter value of the switching amplitude is set as $\mu = 0.5$ for both cases)

4. OPTIMAL DIRECTION OF SWITCHING ACROSS THE STABILITY BOUNDARY

Our results presented in previous sections are consistent with other investigations where either the delay or the coefficient made time dependent stochastically or periodically (18; 19). Our results, however, indicate that the direction we cross the stability boundary may affect the asymptotic stability of the fixed point, and there is the optimal direction for keeping the stability. (Figure 4.) We report here preliminary simulation results that support this conjecture.

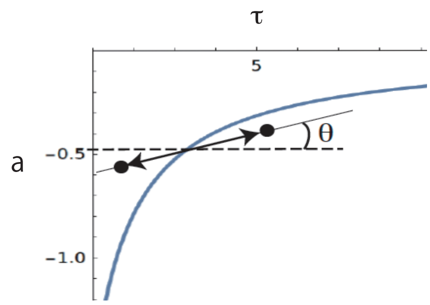

Fig. 4. A schematics of switching across the stability boundary with the angle θ

We extend the basic equation so that we make both the delay and the coefficient time dependent.

$$\frac{dX(t)}{dt} = \hat{a}(t)X(t - \hat{\tau}(t)). \tag{9}$$

As before, we discretize this equation,

$$X(t+1) = X(t) + \hat{a}(t)(dt)X(t - N[\hat{\tau}(t)/dt]), \tag{10}$$

where dt is a time discretization parameter, and $N[s]$ is a function which returns the closest integer to s.

Here, the case of regular alternate switchings is considered so that the values of $(\hat{a}(t), \hat{\tau}(t))$ take the two values alternatively at every time step (i.e., period 2) such that

$$(\hat{a}(t), \hat{\tau}(t)) = \begin{cases} (a_0 + r\sin\theta, \tau_c + r\cos\theta), & (t\ even) \\ (a_0 - r\sin\theta, \tau_c - r\cos\theta), & (t\ odd) \end{cases} \tag{11}$$

In the following, we fix the parameters so that $a_0 = -1.5$, $dt = 0.01$. This leads to the critical delay as $\tau_c \approx 1.0472$ The angle θ is taken in reference with τ axis as in Figure 4.

We show here the representative dynamical path of the (10) by changing the directional angle θ for the two cases of switching amplitudes: $r = 0.1$ (Fig 5) and $r = 0.07$ (Fig 6). We clearly observe that the nature of stability changes as we change the direction of crossing stability boundaries. Even though theoretical understanding of these behaviors are left with future investigations, these are indications that there exists the optimal direction(s) for keeping the asymptotic stability.

5. DISCUSSION

In this paper, we proposed a simple delay differential equation with the switching of delays across the stability boundary. With such a mechanism, the stability region was enhanced in the sense that the stability of the fixed point holds even with the value of the average delay exceeding the normal critical delay value. We have numerically investigated the corresponding discretized map and the phenomena are observed for both alternating and stochastic switching. Analogous switching with the coefficient, however, does not show such enhancement. Thus, how we

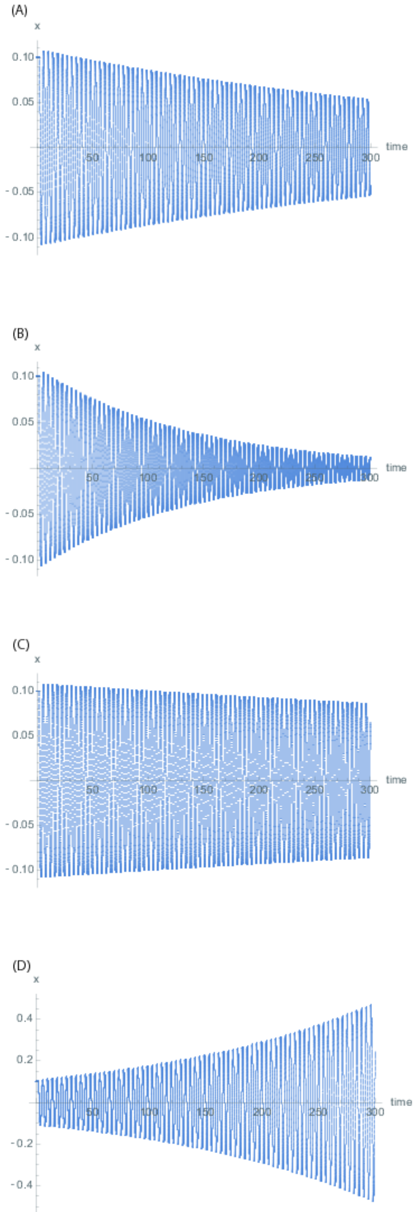

Fig. 5. A representative dynamical path for the equation (10) as we change the crossing angle θ with the switching amplitude $r = 0.1$. (A)$\theta = 0$, (B) $\theta = \pi/8$, (C) $\theta = \pi/4$, (D) $\theta = \pi/2$.

cross the stability boundary by switching mechanism can affect this enhancement phenomenon that was validated by our preliminary simulations. Exploration of the optimal way(s) to cross the stability boundary to have the maximal stability enhancement is an interesting question. As the model is simple, or probably the simplest, it is hoped that some theoretical understanding can be obtained in the future.

We point out again that our results are for the discretized map, and the stability enhancement phenomenon is only inferred from the numerical investigations. Analytical understandings are crucial for this phenomena is real for the differential equations with the delay switching. Further research is needed on this aspect.

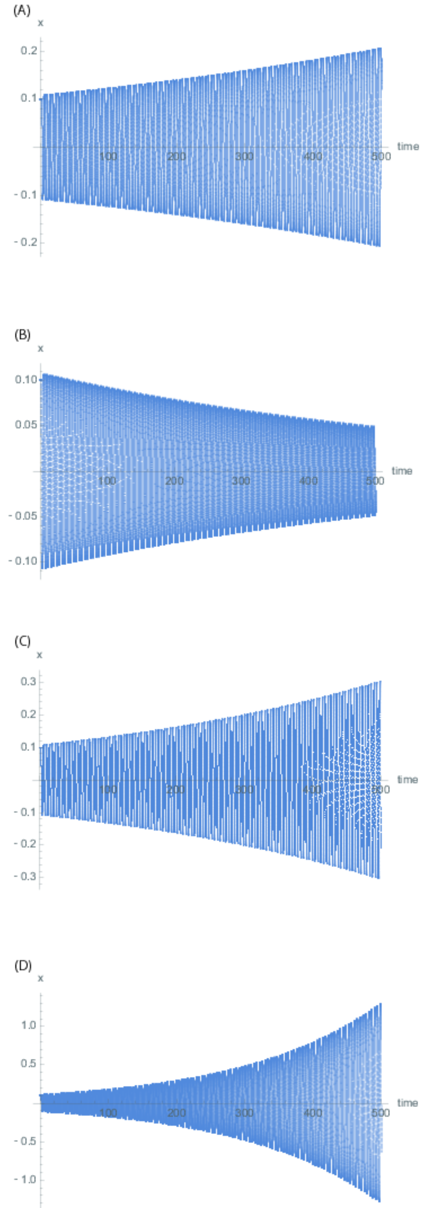

Fig. 6. A representative dynamical path for the equation (10) as we change the crossing angle θ with the switching amplitude $r = 0.07$. (A)$\theta = 0$, (B) $\theta = \pi/8$, (C) $\theta = \pi/4$, (D) $\theta = \pi/2$.

DECLARATION

Author has no conflicts of interest to declare.

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