L-FUNCTIONS WITH RIEMANN'S FUNCTIONAL EQUATION AND THE RIEMANN HYPOTHESIS

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ABSTRACT. Let χ_4 be the non-principal Dirichlet character mod 4 and $L(s,\chi_4)$ be the Dirichlet L-function associated with χ_4, and put $R(s) := s4^s L(s+1,\chi_4) + \pi L(s-1,\chi_4)$. In the present paper, we show that the function $R(s)$ has the Riemann's functional equation and its zeros only at the negative even integers and complex numbers with real part 1/2. We also give other L-functions that have the same property.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Zeta functions with Riemann's functional equation.** Let $\chi(n)$ be a Dirichlet character (mod q). Then, for $\Re(s) := \sigma > 1$, the Riemann zeta function $\zeta(s)$ and the Dirichlet L-function $L(s,\chi)$ are defined by the ordinary Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

The Riemann zeta function $\zeta(s)$ is continued meromorphically and has a simple pole at $s = 1$ with residue 1. The Dirichlet L-function $L(s,\chi)$ can be analytically continued to the whole complex plane to a holomorphic function if $B_0(\chi) := \sum_{r=0}^{q-1} \chi(r)/q = 0$, otherwise to a meromorphic function with a simple pole, at $s = 1$, with residue $B_0(\chi)$. It is well-known that $\zeta(s)$ satisfies Riemann's functional equation

$$\zeta(1-s) = \Gamma_{\cos}(s)\zeta(s), \qquad \Gamma_{\cos}(s) := 2\frac{\Gamma(s)}{(2\pi)^s} \cos\Bigl(\frac{\pi s}{2}\Bigr) \tag{1.1}$$

(e.g., [13, (2.1.8)]). The first converse theorem on $\zeta(s)$ is proved by Hamburger [3, Satz 1] (see also [13, Chapter 2.13]) who characterized $\zeta(s)$ by Riemann's functional equation. Knopp [5] showed that there are infinitely many linearly independent solutions if we relax Hamburger's or Hecke's condition on poles. It should be emphasised that Knopp gives no explicit representation for the solutions satisfying Riemann's functional equation.

For $0 < a \le 1$, we define the Hurwitz zeta function $\zeta(s,a)$ and the periodic zeta function $F(s,a)$ by

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad F(s,a) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}, \qquad \sigma > 1,$$

respectively. The both infinite series of $\zeta(s,a)$ and $F(s,a)$ converge absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this region. The Hurwitz zeta function $\zeta(s,a)$ can be continued for all $s \in \mathbb{C}$ except $s = 1$, where there is a simple pole with residue 1 (e.g., [1, Section 12]). When $0 < a < 1$, the Dirichlet series of $F(s,a)$ converges uniformly in each compact subset of the half-plane $\sigma > 0$ (e.g., [7, p. 20]) and $F(s,a)$ is analytically continuable to the whole complex plane (e.g., [7, Section 2.2]).

2010 *Mathematics Subject Classification.* Primary 11M06, 11M26.

Key words and phrases. L-functions, Riemann's functional equation, Riemann hypothesis.

For $0 < a \le 1/2$, we define the quadrilateral zeta function $Q(s,a)$ by

$$2Q(s,a) := \zeta(s,a) + \zeta(s,1-a) + F(s,a) + F(s,1-a).$$

The function $Q(s,a)$ can be continued analytically to the whole complex plane except $s = 1$. In [9, Theorem 1.1], the author prove the functional equation

$$Q(1-s,a) = \Gamma_{\cos}(s) Q(s,a). \tag{1.2}$$

Note that the gamma factor in (1.2) completely coincides with that of the functional equation for $\zeta(s)$ appearing in (1.1). Moreover, we remark that the functions $\omega_p(s)$ in [2, (4.9)] and $f(s,\chi)$ in [10, Section 2.1] also fulfill Riemann's functional equation (see [2, Section 4.2] and [10, Theorem 2.1]). It should be noted that Riemann's functional equation for $Q(s,a)$, $\omega_p(s)$ and $f(s,\chi)$ do not contradict to Hamburger's theorem since they can not be expressed as any ordinary Dirichlet series.

1.2. **Riemann hypothesis.** From the Euler product of $\zeta(s)$, the Riemann zeta function does not vanish when $\sigma > 1$. In addition, $\zeta(s) \ne 0$ for $\Re(s) < 0$ except for $s = -2n$, where $n \in \mathbb{N}$ by the fact above and the functional equation (1.1). The Riemann hypothesis (RH, in short) is concerned with the locations of nontrivial (non-real) zeros, and states that:

RH *The real part of every nontrivial zero of* $\zeta(s)$ *is* $1/2$.

Let $N(T,\zeta)$ denote the numbers of zeros of $\zeta(s)$ in the region $0 \le \Re(s) \le 1$ and $0 < \Im(s) < T$. Then the following Riemann-von Mangoldt formula is well-known (e.g., [13, Theorem 9.4]). As $T \to \infty$,

$$N(T,\zeta) = \frac{T}{2\pi} \log T - \frac{1 + \log 2\pi}{2\pi} T + O(\log T).$$

It is natural to consider generalizations and analogues of the RH. The generalized Riemann hypothesis extends the RH to all Dirichlet L-functions. More precisely, the generalized Riemann hypothesis asserts that, for every Dirichlet character χ and every complex number $s \notin \mathbb{R}_{<0}$ with $L(s,\chi) = 0$, then the real part of $s \in \mathbb{C}$ is $1/2$.

Both the RH and GRH are unsolved. However, many other examples of L- or zeta functions with analogues of the Riemann hypothesis, some of which have been proved. For instance, Taylor [12] showed that $\zeta^*(s+1/2) - \zeta^*(s-1/2)$, where $\zeta^*(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$, has all its zeros on the critical line $\sigma = 1/2$ (this can be proved by Proposition 2.2). Furthermore, the Riemann hypothesis for some Selberg zeta functions and congruent zeta functions are proved by Selberg and Deligne, respectively (e.g. [14, Section 4]).

Recall that the functions $Q(s,a)$, $\omega_p(s)$ and $f(s,\chi)$ have the Riemann's functional equation. However, the functions $Q(s,a)$ with $a \in \mathbb{R} \setminus \overline{\mathbb{Q}}$ or $a \in \mathbb{Q} \setminus \{1/2, 1/3, 1/4, 1/6\}$ and $f(s,\chi)$ with non-real Dirichlet characters have infinitely many zeros in the both half-plane $\Re(s) > 1$ and vertical strip $1/2 < \Re(s) < 1$ (see [11, Proposition 1.4] and [10, Theorem 2.1 (iv)], respectively). In addition, there are no information on non-real zeros of $\omega_p(s)$ in [2]. On the other hand, zeta functions in the last paragraph satisfy the Riemann hypothesis, but they do not have Riemann's functional equation. Therefore, no one has succeeded to construct L- or zeta functions which satisfy both Riemann's functional equation and the Riemann hypothesis until now.

1.3. **Main results.** Define the functions $R_1(s)$, $R_2(s)$ and $R_3(s)$ by

$$R_1(s) := s3^{s+1} L(s+1,\chi_3) + 2\pi\sqrt{3} L(s-1,\chi_3),$$

$$R_2(s) := s4^{s+1} L(s+1,\chi_4) + 4\pi L(s-1,\chi_4),$$

$$R_3(s) := s(3^{s+1} + 6^{s+1}) L(s+1,\chi_3) + 2\pi\sqrt{3}(1 + 2^{2-s}) L(s-1,\chi_3),$$

where χ_3 and χ_4 are the non-principal Dirichlet characters mod 3 and 4, respectively. Let $N(T,R_j)$ denote the numbers of zeros of $R_j(s)$ in the region $0 \le \Re(s) \le 1$ and $0 < \Im(s) < T$. Then we have the following main result which implies that $R_j(s)$ satisfy both Riemann's functional equation and the Riemann hypothesis for all $j=1,2,3$.

Theorem 1.1. *The function $R_j(s)$ satisfies Riemann's functional equation*

$$R_j(1-s,a) = \Gamma_{\cos}(s) R_j(s), \tag{1.3}$$

has its zeros only at the negative even integers and complex numbers with real part $1/2$ for each $j=1,2,3$. In addition, one has

$$N(T,R_j) = \frac{T}{2\pi}\log T + \frac{\log(r_j^2/2\pi)-1}{2\pi}T + O(\log T), \tag{1.4}$$

where $r_1 = 2$, $r_2 = 3$ and $r_3 = 6$.

The contents of the paper are as follows. In Section 2, we give a proof of Theorem 1.1. In Section 3, we give some numerical calculations for Hardy's Z-functions of $R_j(s)$.

2. Proof

2.1. **Preliminaries.** For $0 < a \le 1/2$, let

$$Z(s,a) := \zeta(s,a)+\zeta(s,1-a), \qquad P(s,a) := F(s,a)+F(s,1-a)$$
$$Y(s,a) := \zeta(s,a)-\zeta(s,1-a), \qquad O(s,a) := -i\bigl(F(s,a)-F(s,1-a)\bigr).$$

Note that $P(s,a)$, $Y(s,a)$ and $O(s,a)$ are entire functions when $0<a<1/2$. However, the function $Z(s,a)$ has a simple pole at $s=1$. Clearly, we have $2Q(s,a) = Z(s,a)+P(s,a)$. In [8, Section 3.1], it is proved that

$$Y(s,1/3) = 3^s L(s,\chi_3), \qquad O(s,1/3) = \sqrt{3} L(s,\chi_3), \tag{2.1}$$
$$Y(s,1/4) = 4^s L(s,\chi_4), \qquad O(s,1/4) = 2L(s,\chi_4), \tag{2.2}$$
$$Y(s,1/6) = (6^s+3^s) L(s,\chi_3), \qquad O(s,1/6) = \sqrt{3}(1+2^{1-s}) L(s,\chi_3). \tag{2.3}$$

Let χ be a real odd primitive character modulo q, where $q>1$, and $G(\chi)$ be the Gauss sum, and put $\varepsilon(\chi) := iq^{-1/2}G(\chi)$. Then, it is widely known (e.g., [1, Exercise 12.8]) that

$$\xi(s,\chi) = \varepsilon(\chi)\xi(1-s,\chi), \qquad \xi(s,\chi) = \left(\frac{q}{\pi}\right)^{(s-1)/2}\Gamma\left(\frac{s-1}{2}\right)L(s,\chi). \tag{2.4}$$

We can easily see that $|L(s,\chi)| \le \zeta(\sigma)$ when $\sigma \ge 3/2$. In addition, we have the following.

Proposition 2.1 ([1, Theorem 12.24])**.** *Let χ be any Dirichlet character modulo q and assume $0<\delta<1$. Then there exists a positive constant $A(\delta)$, depending on δ but not on s or q, such that for $s=\sigma+it$ with $|t|\ge 1$ we have*

$$\bigl|L(s,\chi)\bigr| \le A(\delta)|qt|^{n+1+\delta}, \qquad -n-\delta \le \Re(s) \le -n+\delta, \quad n \in \mathbb{Z}_{\ge -1}.$$

Taylor's theorem mentioned in Section 1.2 can be proved by the following shown by Lagarias and Suzuki [6]. It should be noted that their theorem is a key for the proof of the Riemann hypothesis for $R_j(s)$.

Proposition 2.2 ([6, Theorem 4])**.** *Let $F(s)$ be an entire function of genus zero or one, be real on the real axis, and satisfy $F(s) = \pm F(1-s)$ for some choice of sign, and there exists $\alpha>0$ such that all zeros of $F(s)$ lie in the vertical strip $|\Re(s)-1/2|<\alpha$.*

Then for any real $\gamma \ge \alpha$,

$$\left|\frac{F(s+\gamma)}{F(s-\gamma)}\right| > 1 \quad \text{if} \quad \Re(s) > \frac{1}{2}, \qquad \left|\frac{F(s+\gamma)}{F(s-\gamma)}\right| < 1 \quad \text{if} \quad \Re(s) < \frac{1}{2}.$$

The next proposition is easily proved by the lemma in [13, Section 9.4].

Proposition 2.3. *Let $0 \le \alpha \le \beta < 3$. Let $f(s)$ be an analytic function, real for real s, regular for $\sigma \ge \alpha$; let $|\Re(f(3+it))| \ge m > 0$ and*

$$\left|f(\sigma'+it')\right| < M_{\sigma,t}, \qquad \sigma' \ge \sigma, \quad 1 \le t' \le t.$$

Then if T is not the ordinate of a zero of $f(s)$,

$$\left|\arg f(\sigma+iT)\right| < \frac{\pi}{\log((3-\alpha)/(3-\beta))}\bigl(\log M_{\alpha,T+2} - \log m\bigr) + \frac{3\pi}{2}, \qquad \sigma \ge \beta.$$

2.2. **Proofs of Theorem 1.1.** In order to prove Theorem 1.1, we show the following two lemmas. Note that the statement for real zeros on $R_j(s)$ is easily proved by them.

Lemma 2.4. *For each $j = 1,2,3$, the function $R_j(s)$ satisfies (1.3).*

Lemma 2.5. *For all $j = 1,2,3$, the function $R_j(s)$ do not vanish when $\Re(s) > 1/2$.*

Proof of Lemma 2.4. Clearly, the gamma factor in (1.2) does not depend on $0 < a < 1$. Hence, differentiating the both sides of (1.2) with respect to a, we obtain

$$R(1-s,a) = \Gamma_{\cos}(s) R(s,a), \qquad R(s,a) := 2\frac{\partial}{\partial a} Q(s,a).$$

For $\sigma > 2$, it holds that

$$\frac{\partial}{\partial a} Z(s,a) = \sum_{n=0}^{\infty} \frac{\partial}{\partial a}\left(\frac{1}{(n+a)^s} + \frac{1}{(n+1-a)^s}\right) = -sY(s+1,a),$$

$$\frac{\partial}{\partial a} P(s,a) = \sum_{n=0}^{\infty} \frac{\partial}{\partial a}\frac{\cos(2\pi na)}{n^s} = -2\pi \sum_{n=0}^{\infty} \frac{\sin(2\pi na)}{n^{s-1}} = -2\pi O(s-1,a).$$

Recall that $Y(s,a)$ and $O(s,a)$ are entire functions when $0 < a < 1/2$. Thus, we have

$$\frac{\partial}{\partial a} Z(s,a) + \frac{\partial}{\partial a} P(s,a) = R(s,a) = -sY(s+1,a) - 2\pi O(s-1,a)$$

for all $s \in \mathbb{C}$. From (2.1), (2.2) and (2.3), we have

$$R_1(s) = -R(s,1/3), \qquad R_2(s) = -R(s,1/4), \qquad R_3(s) = -R(s,1/6).$$

Therefore, we obtain Riemann's functional equation for $R_j(s)$. □

Proof of Lemma 2.4. First consider the case $j = 1$. Then one has $\varepsilon(\chi_3) = -1$ from $G(\chi_3) = i\sqrt{3}$. In this case, we have

$$\xi(s,\chi_3) = -\xi(1-s,\chi_3), \qquad \xi(s,\chi_3) = \left(\frac{3}{\pi}\right)^{(s-1)/2} \Gamma\left(\frac{s-1}{2}\right) L(s,\chi_3)$$

from (2.4). Clearly, by the definition of $\xi(s,\chi_3)$, one has

$$\xi(s+1,\chi_3) = \left(\frac{3}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s+1,\chi_3) = \left(\frac{3}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}-1\right) \frac{s}{2}\, L(s+1,\chi_3),$$

$$\xi(s-1,\chi_3) = \left(\frac{3}{\pi}\right)^{s/2-1} \Gamma\left(\frac{s}{2}-1\right) L(s-1,\chi_3) = \left(\frac{3}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}-1\right) \frac{\pi}{3}\, L(s-1,\chi_3).$$

We can see that $\xi(s,\chi_3)$ is an entire function of genus one and does not vanish when $\Re(s) > 1$ by the Hadamard product factorization of $\xi(s,\chi_3)$ and the Euler product of $L(s,\chi_3)$, respectively. Hence, from the equations above and Proposition 2.2, we have

$$\left|3^{1+1/2} s L(s+1,\chi_3)\right| > \left|2\pi\sqrt{3} L(s-1,\chi_3)\right|, \qquad \Re(s) > 1/2.$$

Obviously, $|3^{1+s}| > |3^{1+1/2}|$ when $\Re(s) > 1/2$. Hence, we obtain

$$\bigl|s3^{s+1}L(s+1,\chi_3)\bigr| > \bigl|2\pi\sqrt{3}L(s-1,\chi_3)\bigr|, \qquad \Re(s) > 1/2. \tag{2.5}$$

This inequality implies that $R_1(s)$ does not vanish if $\Re(s) > 1/2$.

Similarly, we can prove the case $j = 2$. We have

$$\xi(s,\chi_4) = -\xi(1-s,\chi_4), \qquad \xi(s,\chi_4) = \left(\frac{4}{\pi}\right)^{(s-1)/2}\Gamma\left(\frac{s-1}{2}\right)L(s,\chi_4)$$

from $G(\chi_4) = 2i$. By the definition of $\xi(s,\chi_4)$, we have

$$\xi(s+1,\chi_4) = \left(\frac{4}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)L(s+1,\chi_4) = \left(\frac{4}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}-1\right)\frac{s}{2}\,L(s+1,\chi_4),$$

$$\xi(s-1,\chi_4) = \left(\frac{4}{\pi}\right)^{s/2-1}\Gamma\left(\frac{s}{2}-1\right)L(s-1,\chi_4) = \left(\frac{4}{\pi}\right)^{s/2}\Gamma\left(\frac{s}{2}-1\right)\frac{\pi}{4}\,L(s-1,\chi_4).$$

By Proposition 2.2 again, we have

$$\bigl|8sL(s+1,\chi_4)\bigr| > \bigl|4\pi L(s-1,\chi_4)\bigr|, \qquad \Re(s) > 1/2.$$

The inequality above and $|4^{1+s}| > 8$ with $\Re(s) > 1/2$ imply the RH for $R_2(s)$.

The case $j = 3$ is proved by (2.5). Taking into account the triangle with vertices 0, $1 + 2^{2-\overline{s}}$ and $1 + 2^{1+s}$, where $\overline{s}$ is the complex conjugate of s, we can see that $|1 + 2^{s+1}| > |1 + 2^{2-\overline{s}}|$ which implies $|1 + 2^{s+1}| > |1 + 2^{2-s}|$ when $\Re(s) > 1/2$ (see also the proof of [8, Proposition 1.8]). Hence, by (2.5),

$$\begin{aligned}\bigl|s(3^{s+1} + 6^{s+1})L(s+1,\chi_3)\bigr| &= \bigl|(1 + 2^{s+1})s3^{s+1}L(s+1,\chi_3)\bigr| \\ &> \bigl|(1 + 2^{s+1})2\pi\sqrt{3}L(s-1,\chi_3)\bigr| > \bigl|2\pi\sqrt{3}(1 + 2^{2-s})L(s-1,\chi_3)\bigr|\end{aligned}$$

which implies that $R_3(s)$ does not vanish when $\Re(s) > 1/2$. □

Proof of Theorem 1.1. We only have to show the Riemann-von Mangoldt formula (1.4). Let $j = 1$ and $R_1(s) = s3^{s+1}R_1^*(s)$. From the argument in the proof of [13, Theorem 9.3], functional equations (1.1) and (1.3), we have

$$\pi N(T,R_1) = \Delta \arg s(s-1) + \Delta \arg \pi^{-s/2} + \Delta \arg \Gamma(s/2) + \Delta \arg s3^{s+1} + \Delta \arg R_1^*(s),$$

where Δ denotes the variation from 3 to $3+iT$, and then to $1/2+iT$, along straight lines. By the estimations in the proof of [13, Theorem 9.3], we obtain

$$\begin{aligned}&\Delta \arg s(s-1) + \Delta \arg \pi^{-s/2} + \Delta \arg \Gamma(s/2) + \Delta \arg s3^{s+1} \\ =&\frac{T}{2}\log\frac{T}{2} - \frac{T}{2} - \frac{T}{2}\log\pi + T\log 3 + O(1).\end{aligned}$$

Now we consider $\Delta \arg R_1^*(s)$. Clearly, one has

$$\bigl|\Re\bigl(R_1^*(3+it)\bigr)\bigr| > 1 - \bigl(\zeta(4) - 1\bigr) - 2\pi 3^{1/2-4}\bigl|L(2,\chi_3)\bigr| > 2 - \zeta(4) - 2\pi 3^{-7/2}\zeta(2) > 0.69.$$

Applying Proposition 2.3 with $f(s) = R_1^*(s)$, $\alpha = 1/4$ and $\beta = 1/2$, we obtain

$$\Delta \arg R_1^*(s) = O(\log T)$$

by Proposition 2.1. Therefore, we have (1.4) for $R_1(s)$. Similarly, we can show the cases $j = 2$ and $j = 3$. □

3. Numerical calculations

3.1. Hardy's *Z*-function. From [4, (1.6) and (1.7)], Riemann's functional equation for $\zeta(s)$ can be rewritten as

$$\zeta(s) = \eta(s)\zeta(1-s), \qquad \eta(s) := \frac{1}{\Gamma_{\cos}(s)} = \frac{\Gamma(1/2 - s/2)}{\Gamma(s/2)}\pi^{s-1/2}.$$

Using $\eta(s)$ above, we define Hardy's Z-function $Z(t)$ by

$$Z(t) := \bigl(\eta(1/2+it)\bigr)^{-1/2}\zeta(1/2+it) = e^{i\theta(t)}\zeta(1/2+it),$$

where $\theta(t) := \Im(\log\Gamma(1/4+it/2)) - (t/2)\log\pi$. It is widely-known (e.g. [4, Chapter 1.3]) that for $t \in \mathbb{R}$,

$$Z(t) \in \mathbb{R}, \qquad |Z(t)| = |\zeta(1/2+it)|, \qquad Z(t) = Z(-t).$$

Since $R_j(s)$, where $j = 1,2,3$, satisfy Riemann's functional equation, we can define

$$H_j(t) := e^{i\theta(t)}R_j(1/2+it), \qquad j = 1,2,3$$

as an analogue of $Z(t)$. By modifying the argument in [4, Chapter 1.3], we have

$$H_j(t) \in \mathbb{R}, \qquad |H_j(t)| = |R_j(1/2+it)|, \qquad H_j(t) = H_j(-t).$$

3.2. Figures. All figures are given by Mathematica 13.0. Note that they are plotted by not $(t+1)^{-1}H_j(1/2+it)$ but $\Re((t+1)^{-1}H_j(1/2+it))$ since Mathematica 13.0 can not regard $H_j(1/2+it)$ as real functions.

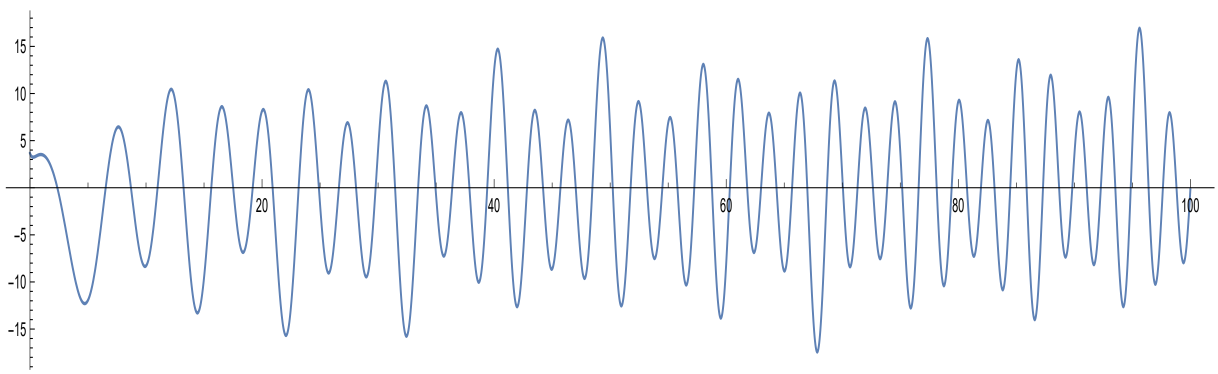

Figure 1. $\{(t+1)^{-1}H_1(1/2+it) : 0 \le t \le 100\}$

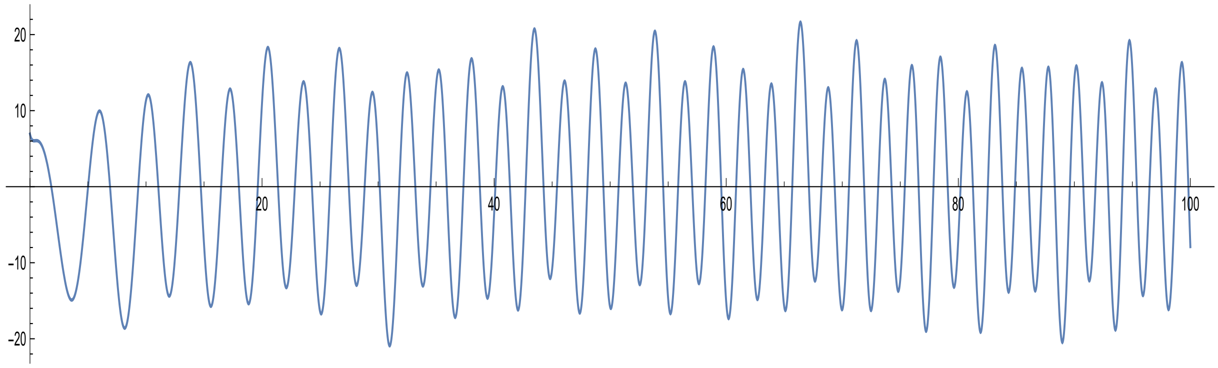

Figure 2. $\{(t+1)^{-1}H_2(1/2+it) : 0 \le t \le 100\}$

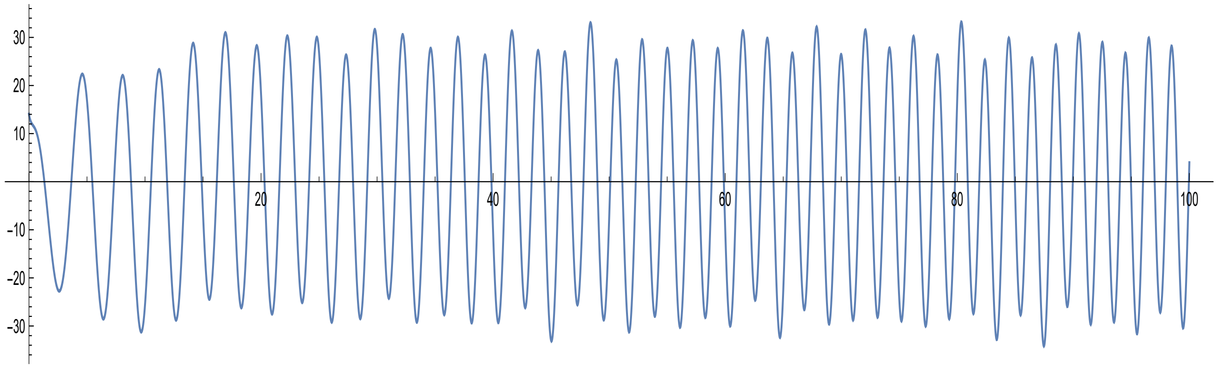

FIGURE 3. $\{(t+1)^{-1}H_3(1/2+it) : 0 \le t \le 100\}$

Acknowledgments. The author was partially supported by JSPS grant 16K05077.

Conflict of Interest. The authors have no conflicts of interest directly relevant to the content of this article.

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