Generative Operators of Natural Informational Forces: A Unified Framework for MI-based Dynamics

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Abstract

Classical formulations typically describe natural forces in terms of potentials, vector fields, or variational principles that dictate the dynamics of physical systems. However, recent advances in information-theoretic approaches suggest a more general characterization. Specifically, it is posited that forces can be fundamentally understood by their role in reshaping probability distributions and altering informational dependencies among interacting systems.

In this work, we present a unified theoretical framework in which natural forces are characterized as specific components of generating operators that act on probability densities. In the context of a Markovian evolution described by the differential equation

$$\partial_t p_t = L p_t,$$

we identify the force-associated component, denoted L_{att}, as the canonical component of the operator that monotonically contracts the information radius defined by

$$r(t) = H(X_t) + H(Y_t) - 2\,\mathrm{MI}(X_t; Y_t).$$

We also demonstrate that this contraction property precisely corresponds to the mutual information gradient flow.

This operator-theoretic interpretation extends the concept of natural forces beyond classical geometric constructs to generators of informational coupling. Thus, it provides a rigorous, unified perspective that integrates interaction theories in physics, divergence contraction in information geometry, and the theory of Markov semigroups. The resulting framework offers a coherent, coordinate-free description of natural forces as operator components responsible for the creation, preservation, or enhancement of mutual information, independent of any privileged temporal or spatial parametrization.

Keywords: natural forces; generating operators; mutual information gradient flow; information radius; Markov semigroups

1 Introduction

Classical accounts of natural forces rely on potentials, vector fields, and variational principles that determine the dynamics of physical systems [1, 2]. While this geometric–mechanical picture remains foundational, recent information-theoretic perspectives have revealed that interactions can be fruitfully characterized by how they reshape probability distributions and reorganize informational dependencies among coupled systems [3, 4, 5]. This observation motivates an operator-theoretic reformulation: rather than treating forces solely as fields on a configuration space, we study them as components of *generating operators* that govern the time evolution of probability densities [6, 7].

Background and motivation. Let p_t denote the joint density of two interacting systems (X_t, Y_t) evolving under a Markovian dynamics

$$\partial_t p_t \;=\; L\, p_t, \tag{1.1}$$

where L is the generator of a (sub-)Markov semigroup [8, 6]. In prior work, we established contraction principles for the *information radius*

$$r(t) \;=\; H(X_t) + H(Y_t) \;-\; 2\,\mathrm{MI}(X_t; Y_t), \tag{1.2}$$

under flows aligned with the mutual-information (MI) gradient [17, 18]. Equivalently, continuous evolution in the direction of increasing MI yields a monotone decrease of r [17]. Complementing these variational statements, a velocity-field decomposition of the form

$$v_x \;=\; \gamma\, \nabla_x \phi \;+\; u_x, \qquad v_y \;=\; \gamma\, \nabla_y \phi \;+\; u_y, \tag{1.3}$$

separates the MI-generating component (controlled by a scalar potential ϕ and gain $\gamma > 0$) from marginal drifts u_x, u_y [18]. Taken together, these results suggest that the *force-carrying* part of the evolution is precisely the component that contracts r.

Problem addressed. The central question of this paper is to formalize the preceding intuition at the level of the generator L. Specifically, we seek a principled operator decomposition

$$L \;=\; L_{\mathrm{att}} \;+\; L_{\mathrm{res}}, \tag{1.4}$$

in which the *attractive* component L_{att} is characterized by a certified contraction of r along (1.1), while the residual part L_{res} is neutral with respect to r (e.g., preserving marginals or representing symmetries and transports that do not create informational coupling).

Our contributions. This work develops a unified operator-theoretic framework for natural forces with the following contributions:

1. **Operator characterization of natural forces.** Natural forces are defined as those components of the generator that *monotonically contract* the information radius (1.2) along solutions of (1.1) [17, 18].

2. **Equivalence with MI-gradient generation.** We prove that the contraction property is *equivalent* to the generation of MI-gradient flow [17].

3. **Structural decomposition and neutrality criteria.** Building on the velocity-field decomposition (1.3), we provide conditions under which L_{res} is neutral with respect to r [18].

4. **Conceptual unification.** The framework links interaction theories in physics, divergence contraction in information geometry, and Markov semigroup theory [5, 6, 8].

Scope and implications. In particular, the notion of "force" used here is informational or entropic in nature: it does not refer to a mechanical force in the Newtonian sense, but to the component of the generator that increases mutual information and contracts r. By elevating forces from geometric fields to operator components on probability spaces, the proposed framework focuses on invariant informational effects (creation, preservation, and enhancement of mutual information) [4, 5]. This shift clarifies when an interaction is genuinely *force-like* in the informational sense and provides operator-level criteria to detect and quantify such effects .

Organization. Section 2 reviews variational identities and r. Section 3 introduces the generator-level decomposition and contraction criteria. Section 4 proves the equivalence between r-contraction and MI-gradient generation and records neutrality and stability results. Section 5 develops structural and flux-based decompositions. Section 6 presents applications and model-specific consequences. Section 7 discusses implications and possible extensions.

2 Preliminaries

This section fixes notation, recalls the information-theoretic functionals used throughout, and states the variational identities that will be employed in later sections. We also summarize the generator formalism for Markovian evolutions and the velocity-field decomposition derived in prior work.

2.1 Notation and setting

Let $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ be Polish spaces endowed with reference measures (e.g. Lebesgue on $\mathbb{R}^{d_x}$ and $\mathbb{R}^{d_y}$). We consider a joint density $p_t(x, y)$ of two interacting systems (X_t, Y_t) evolving according to a (sub-)Markov semigroup $\{P_t\}_{t\geq 0}$ with infinitesimal generator L, so that

$$\partial_t p_t = L\, p_t, \qquad p_t = P_t p_0, \tag{2.1}$$

in the weak (distributional) sense [8, 6, 7]. We write $p_t^X(x) = \int p_t(x, y)\, dy$ and $p_t^Y(y) = \int p_t(x, y)\, dx$ for the marginals, and we use $\langle f, g\rangle = \int f\, g$ to denote the L^2 pairing whenever well-defined.

2.2 Information functionals

We adopt Shannon entropy and mutual information as the basic primitives [3, 4, 5]:

$$H(X_t) = -\int p_t^X(x)\, \log p_t^X(x)\, dx, \tag{2.2}$$

$$H(Y_t) = -\int p_t^Y(y)\, \log p_t^Y(y)\, dy, \tag{2.3}$$

$$\mathrm{MI}(X_t; Y_t) = \int p_t(x, y)\, \log \frac{p_t(x, y)}{p_t^X(x)\, p_t^Y(y)}\, dx\, dy. \tag{2.4}$$

The *information radius* employed in this work is

$$r(t) = H(X_t) + H(Y_t) - 2\,\mathrm{MI}(X_t; Y_t), \tag{2.5}$$

which decreases when MI increases, under suitable regularity [17, 18]. While r is not a metric in the axiomatic sense, it serves as a Lyapunov-type functional for the evolutions considered here and admits well-defined first variations under mild integrability assumptions [5, 4].

2.3 Generator calculus and variational derivatives

For a sufficiently regular functional $\mathcal{F}[p]$ on densities, its time derivative along the evolution (2.1) can be expressed using the Gâteaux derivative $\delta\mathcal{F}/\delta p$:

$$\frac{d}{dt}\,\mathcal{F}[p_t] = \Big\langle \frac{\delta\mathcal{F}}{\delta p}(p_t),\, \partial_t p_t \Big\rangle = \Big\langle \frac{\delta\mathcal{F}}{\delta p}(p_t),\, Lp_t \Big\rangle = \big\langle L^* \tfrac{\delta\mathcal{F}}{\delta p}(p_t),\, p_t \big\rangle, \tag{2.6}$$

where L^* is the L^2-adjoint (the Fokker–Planck operator when L is a Kolmogorov generator) [8, 6]. Applying (2.6) to $\mathcal{F} = H(X)$ and $H(Y)$ yields

$$\frac{d}{dt}H(X_t) = -\int \big(1 + \log p_t^X(x)\big)\, \partial_t p_t^X(x)\, dx, \tag{2.7}$$

$$\frac{d}{dt}H(Y_t) = -\int \big(1 + \log p_t^Y(y)\big)\, \partial_t p_t^Y(y)\, dy, \tag{2.8}$$

with $\partial_t p_t^X = \int Lp_t\, dy$ and $\partial_t p_t^Y = \int Lp_t\, dx$. For MI, using $\delta\mathrm{MI}/\delta p = \log \frac{p}{p^X p^Y}$ (under a fixed reference measure) gives

$$\frac{d}{dt}\,\mathrm{MI}(X_t;Y_t) = \int \log \frac{p_t(x,y)}{p_t^X(x)p_t^Y(y)}\, \big(Lp_t\big)(x,y)\, dx\, dy, \tag{2.9}$$

whenever the interchange of differentiation and integration is justified [4, 5]. Combining these identities yields a general expression for $\frac{d}{dt}r(t)$ in terms of L and (p_t, p_t^X, p_t^Y) [17].

Variational identity for r. Collecting the previous formulas, we will repeatedly use the compact pairing notation

$$\frac{d}{dt}\,r(t) = \big\langle \nabla r(t),\, Lp_t \big\rangle = \big\langle \nabla r(t),\, L_{\mathrm{att}} p_t \big\rangle + \big\langle \nabla r(t),\, L_{\mathrm{res}} p_t \big\rangle, \tag{2.10}$$

whenever the splitting $L = L_{\mathrm{att}} + L_{\mathrm{res}}$ is specified. Here ∇r denotes the variational derivative of r with respect to p, understood via the adjoint representation in (2.6) [5, 17].

2.4 Velocity-field decomposition (recall)

For drift–diffusion dynamics on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$, the probability flux can be written as $(p_t v_x, p_t v_y)$, and prior work established the decomposition

$$v_x = \gamma\, \nabla_x \phi + u_x, \qquad v_y = \gamma\, \nabla_y \phi + u_y, \qquad \gamma > 0, \tag{2.11}$$

Throughout this paper, the gain parameter $\gamma > 0$ is treated as a scalar constant; allowing $\gamma = \gamma(x,y)$ is possible but would require additional regularity assumptions that do not affect the contraction arguments below.

where the potential part generates increase of MI (and hence decrease of r), while the residual drifts u_x, u_y preserve marginals to first order and are neutral with respect to r under suitable constraints [18, 17]. The decomposition (2.11) underlies the operator splitting $L = L_{\mathrm{att}} + L_{\mathrm{res}}$ used in Sections 3–5.

2.5 Standing assumptions

We summarize the regularity conditions assumed throughout:

1. **Well-posedness.** The semigroup $\{P_t\}_{t\geq 0}$ is Feller (or sub-Markov) with generator L densely defined on a core; p_t exists and is unique for $t \in [0,T]$ for any $T > 0$ [8, 6].

2. **Integrability and smoothness.** Densities and marginals are strictly positive and admit logarithms in L^1_{loc}; differentiation under the integral sign is valid for $H(X_t)$, $H(Y_t)$, and $\text{MI}(X_t; Y_t)$ [4, 5].
3. **Boundary behavior.** Either compact support with reflecting boundaries or sufficiently fast decay ensures that boundary terms vanish in integrations by parts (when L is of Kolmogorov type) [6].
4. **Neutrality constraints.** When invoked, the residual component L_{res} preserves marginals to first order and does not increase r; precise neutrality criteria are given later in Section 5 [18].

Support condition. We assume that the support of p_t coincides with that of $p_t^X \otimes p_t^Y$, so that the mutual-information potential $\phi_t = \log \frac{p_t}{p_t^X p_t^Y}$ is finite almost everywhere.

A basic contraction lemma. Under the standing assumptions and the splitting $L = L_{\text{att}} + L_{\text{res}}$ with L_{res} neutral, we will employ the following lemma (proof deferred to Appendix A):

Lemma 2.1. *If $\langle \nabla r(t), L_{\text{att}} p_t \rangle \leq 0$ for all $t \in [0, T]$, then $r(t)$ is nonincreasing on $[0, T]$. Moreover, strict negativity on a set of times of positive measure implies $r(t)$ is strictly decreasing on that set.*

The remainder of the paper leverages the identities in this section to obtain an operator-level characterization of natural forces (Section 3), an equivalence with MI-gradient generation (Section 4), and neutrality criteria linked to the velocity-field structure (Section 5) [17, 18].

3 Operator-Level Characterization of Natural Forces

In this section, we formalize the operator-theoretic identification of natural forces. Building on the variational calculus and the velocity-field decomposition established in Section 2, we define a canonical splitting of the generator L into an attractive, force-generating part and a residual, neutral part. We then state and prove the main contraction theorem for the information radius.

3.1 Operator splitting: attractive and residual components

Given the Markovian evolution

$$\partial_t p_t = L\, p_t, \tag{3.1}$$

we consider a decomposition of the generator of the form

$$L = L_{\text{att}} + L_{\text{res}}, \tag{3.2}$$

where L_{att} is responsible for the monotonic contraction of the information radius, and L_{res} is neutral in the sense that it does not change r.

Definition (Attractive component). We call L_{att} an *attractive (force-carrying) component* of L if, for any admissible density p,

$$\big\langle \nabla r(p),\, L_{\text{att}} p \big\rangle \leq 0. \tag{3.3}$$

The residual component is then defined by $L_{\mathrm{res}} = L - L_{\mathrm{att}}$ and is *neutral* if

$$\big\langle \nabla r(p),\, L_{\mathrm{res}} p \big\rangle = 0 \quad \text{for all admissible } p, \tag{3.4}$$

(with the weaker form ≤ 0 admissible when only nonincrease is required).[1]

3.2 Main theorem: operator contraction principle

In what follows, "gradient flow" refers to steepest ascent/descent with respect to a fixed metric g_p on the statistical manifold (e.g., Fisher–Rao or a Wasserstein-type metric) under which ∇r is computed.

Theorem 3.1 (Operator Contraction Principle)**.** *Let p_t evolve according to* (3.1) *with generator $L = L_{\mathrm{att}} + L_{\mathrm{res}}$ as above, and suppose the standing assumptions in Section 2.5 hold. Then the following are equivalent:*

(i) $\dfrac{d}{dt} r(t) = \big\langle \nabla r(t),\, L_{\mathrm{att}} p_t \big\rangle \leq 0$ *for all* $t \in [0, T]$*;*

(ii) The flow generated by L_{att} coincides with the direction of steepest ascent of mutual information, i.e., L_{att} generates the MI-gradient flow.

Moreover, strict negativity of (i) *on a set of positive measure implies strict contraction of r on that set.*

Proof sketch. The claim follows from the variational identities in Section 2.3, the splitting construction (3.2), and the equivalence between MI-gradient directions and r contraction established in prior work. The metric g_p fixes the Riesz representation of variational derivatives, aligning the pairing in (3.3) with gradient ascent of MI.

3.3 Examples

Example (Coupled Ornstein–Uhlenbeck). Consider the joint evolution of (X_t, Y_t) under coupled linear drift and independent diffusion:

$$\begin{aligned} dX_t &= (-\alpha X_t + \beta Y_t)\, dt + \sqrt{2D}\, dW_t^X, \\ dY_t &= (-\gamma Y_t + \delta X_t)\, dt + \sqrt{2D}\, dW_t^Y, \end{aligned}$$

where W_t^X, W_t^Y are independent Brownian motions. The generator L is quadratic; the attractive component L_{att} collects the cross-drift terms that increase MI. Under suitable spectral positivity of the cross-drift block and coercivity of the symmetric part of L, the information radius r contracts monotonically.

Example (Information-geometric gradient flow). Let

$$L_{\mathrm{att}} p \;=\; -\nabla\cdot\Big(p\, \nabla \log \frac{p}{p^X p^Y}\Big),$$

the Fokker–Planck form of the MI-gradient flow on the product space. Here L_{att} generates monotone contraction of r, while L_{res} may include symmetries or transports that preserve MI. Well-posedness requires regularity and nondegeneracy ensuring p^X, p^Y are strictly positive and sufficiently smooth.

[1] When L_{att} is chosen p-dependently, the resulting evolution $\partial_t p = L_{\mathrm{att}}[p]\, p$ becomes nonlinear; our variational statements remain valid and will be explicitly phrased at the level of the induced pairings.

3.4 Remarks on uniqueness and structural constraints

The splitting $L = L_{\text{att}} + L_{\text{res}}$ is generally not unique without additional structure. For drift–diffusion models with the velocity-field decomposition of Section 2.4, the MI-generating direction determines L_{att} uniquely. In more general Markovian systems, neutrality or symmetry constraints can be imposed to specify L_{res} canonically.

This completes the operator-level formulation of natural forces. The next section establishes the full equivalence between operator contraction, MI-gradient generation, and information-theoretic dynamics under general conditions.

4 Equivalence, Neutrality, and Stability Results

This section develops equivalences between operator-level contraction, mutual-information (MI) gradient generation, and dissipation inequalities for the information radius. We also formalize neutrality criteria for the residual component and record stability properties under perturbations of the generator. Throughout, the standing assumptions of Section 2.5 are in force, and gradients are taken with respect to a fixed metric g_p on the statistical manifold (e.g., Fisher–Rao or a Wasserstein-type metric) [5, 6].

4.1 Equivalent characterizations of contraction

Let p_t solve $\partial_t p_t = L p_t$ with the splitting $L = L_{\text{att}} + L_{\text{res}}$ introduced in Section 3. The following theorem gathers three equivalent ways to express informational contraction.

Theorem 4.1 (Three-way equivalence)**.** *The following statements are equivalent on any interval* $[0, T]$*:*

(i) ***Operator contraction:*** *for all* t, $\dfrac{d}{dt} r(t) = \langle \nabla r(t),\, L_{\text{att}} p_t \rangle \le 0.$

(ii) ***MI-gradient generation:*** *the direction of steepest ascent of* $\mathrm{MI}(X_t; Y_t)$ *with respect to* g_{p_t} *is generated by* L_{att}*, i.e.* L_{att} *induces the MI-gradient flow [17, 18].*

(iii) ***Dissipation inequality (EVI-type):*** *there exists* $\kappa \ge 0$ *such that for all* $s < t$

$$r(t) - r(s) \;\le\; -\int_s^t \big\|\nabla r(\tau)\big\|^2_{g_{p_\tau}}\, d\tau + \kappa \int_s^t \Psi(p_\tau)\, d\tau, \tag{4.1}$$

where Ψ *collects neutrality/symmetry terms induced by* L_{res} *and vanishes when* L_{res} *is neutral.*

If, in addition, L_{res} *is neutral in the sense of* (3.4)*, then one may take* $\kappa = 0$ *and* (4.1) *reduces to the strict dissipation inequality.*

Proof sketch. (i)⇒(ii): By Riesz representation under g_{p_t}, the variational identity $\frac{d}{dt} r(t) = \langle \nabla r(t), L_{\text{att}} p_t \rangle$ implies that $L_{\text{att}} p_t$ aligns with the g_{p_t}-gradient direction of $-r$ and hence generates the ascent of MI (equivalently, the descent of r) [5, 17]. (ii)⇒(iii) follows from standard energy–dissipation arguments for gradient flows [6], while (iii)⇒(i) is obtained by differentiation at the integrand level and neutrality of L_{res}. Full details are given in Appendix A. □

4.2 Neutrality and symmetry

We record operator-level conditions that ensure neutrality of L_{res}. Let Π_X, Π_Y denote the marginalization operators $\Pi_X p = \int p(\cdot, y)\,dy$ and $\Pi_Y p = \int p(x, \cdot)\,dx$.

Proposition 4.2 (Neutrality via marginal preservation)**.** *If L_{res} preserves marginals to first order, i.e. $\partial_t p_t^X = \Pi_X L_{\mathrm{res}} p_t = 0$ and $\partial_t p_t^Y = \Pi_Y L_{\mathrm{res}} p_t = 0$ for all t, and if L_{res} commutes with the projections induced by (Π_X, Π_Y) in the sense that*

$$\big\langle \log(p_t^X) + \log(p_t^Y),\, L_{\mathrm{res}} p_t \big\rangle = 0, \tag{4.2}$$

then L_{res} is neutral with respect to r, i.e. $\langle \nabla r(t), L_{\mathrm{res}} p_t \rangle = 0$ for all t.

Proof idea. Differentiate $H(X_t)$ and $H(Y_t)$ along L_{res} using the chain rule (2.6) and invoke (4.2) to cancel contributions from the marginals; the remaining term corresponds to MI and vanishes under marginal preservation and commutation. See Appendix B for details. □

Proposition 4.3 (Symmetry-based neutrality)**.** *Suppose L_{res} is anti-self-adjoint with respect to the pairing induced by g_p on the subspace tangent to isomarginal deformations and preserves the product structure $p^X p^Y$. Then L_{res} is neutral.*

Proof idea. Anti-self-adjointness implies orthogonality of $L_{\mathrm{res}} p$ to $\nabla r(p)$ in the g_p-inner product. Preservation of $p^X p^Y$ eliminates spurious mutual-information production. Details are provided in Appendix B. □

4.3 Stability under perturbations

We next quantify robustness of contraction under bounded perturbations of L_{att}.

Theorem 4.4 (Perturbation stability)**.** *Let $\widetilde{L}_{\mathrm{att}} = L_{\mathrm{att}} + K$ with a linear perturbation K satisfying, for some $\eta \in [0, 1)$,*

$$\big|\big\langle \nabla r(p),\, Kp \big\rangle\big| \;\le\; \eta\, \big\|\nabla r(p)\big\|_{g_p}^2 \qquad \text{for all admissible } p. \tag{4.3}$$

If L_{res} is neutral, then the evolution driven by $\widetilde{L}_{\mathrm{att}} + L_{\mathrm{res}}$ still contracts r and obeys

$$\frac{d}{dt} r(t) \;\le\; -(1-\eta)\, \big\|\nabla r(t)\big\|_{g_{p_t}}^2. \tag{4.4}$$

In particular, Grönwall's lemma yields exponential decay when $\|\nabla r(t)\|_{g_{p_t}}^2 \ge \lambda\big(r(t) - r^\star\big)$ for some $\lambda > 0$ and equilibrium value $r^\star$.

Proof sketch. Combine (3.3) with the bound (4.3) and neutrality of L_{res} to obtain (4.4); the last claim follows by a standard differential inequality argument [6]. □

4.4 Discrete-time semigroup version

Let $\{P_h\}_{h>0}$ be a family of Markov operators with $p_{k+1} = P_h p_k$ and generator $L_h = (P_h - I)/h$. If $L_h = L_{\mathrm{att},h} + L_{\mathrm{res},h}$ with the discrete neutrality $\langle \nabla r_k, L_{\mathrm{res},h} p_k \rangle = 0$, and if

$$r(P_h p) - r(p) \;\le\; -h\, \big\|\nabla r(p)\big\|_{g_p}^2 + o(h), \tag{4.5}$$

then r_k is nonincreasing in k, and the continuous-time results are recovered as $h \to 0$ [8].

Summary. The results above establish a robust equivalence between operator-level contraction, MI-gradient generation, and dissipation inequalities; provide verifiable neutrality criteria; and show stability under perturbations. These tools will be used in Section 5 to analyze structural decompositions linked to the velocity-field representation and to derive model-specific consequences.

5 Structural Decompositions and Velocity-Field Representation

This section develops structural decompositions that connect the operator splitting $L = L_{\text{att}} + L_{\text{res}}$ with velocity-field representations of the probability flux. We provide a canonical construction of the attractive direction from a variational projection, clarify neutrality on the level of fluxes, and record uniqueness criteria under regularity and boundary conditions. Throughout we refer to the preliminaries of Section 2 and adopt the same metric g_p for gradients [5, 6].

5.1 Flux form and Helmholtz-type decomposition

For drift–diffusion dynamics on $\mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$, the Fokker–Planck form reads $\partial_t p_t + \nabla\cdot J_t = 0$ with flux $J_t = p_t\, v_t - D\nabla p_t$ (with a suitable diffusion tensor D) [6]. Under mild regularity and decay or reflecting boundaries, one has the orthogonal decomposition

$$v_t \;=\; v_t^{\parallel} \;+\; v_t^{\perp}, \qquad \text{with} \quad v_t^{\parallel} \in \overline{\{\nabla\phi : \phi \in C^\infty\}}, \quad \nabla\cdot(p_t v_t^{\perp}) = 0, \tag{5.1}$$

where orthogonality is taken in the $L^2(p_t)$ inner product [6]. The component $v_t^{\parallel}$ is the unique (up to constants) gradient field that best approximates v_t in $L^2(p_t)$; $v_t^{\perp}$ is solenoidal with respect to the weighted divergence $\nabla\cdot(p_t\cdot)$ and generates transports that preserve certain integral observables.

Attractive direction from MI potential. Let $\phi_t := \log\frac{p_t}{p_t^X p_t^Y}$ be the instantaneous MI potential. The MI-gradient velocity is $v_t^{\text{MI}} := \gamma\nabla\phi_t$ with gain $\gamma > 0$. Projecting v_t onto the span of $\nabla\phi_t$ in $L^2(p_t)$ yields the canonical attractive direction

$$v_t^{\text{att}} \;:=\; \text{Proj}^{L^2(p_t)}_{\text{span}\{\nabla\phi_t\}}(v_t) \;=\; \frac{\langle v_t, \nabla\phi_t\rangle_{L^2(p_t)}}{\|\nabla\phi_t\|^2_{L^2(p_t)}}\,\nabla\phi_t, \tag{5.2}$$

which maximally increases MI (equivalently, decreases r) among rank-one gradient directions determined by ϕ_t [17, 18]. The residual velocity $v_t^{\text{res}} := v_t - v_t^{\text{att}}$ is $L^2(p_t)$-orthogonal to $\nabla\phi_t$ and is neutral to first order.

5.2 Operator realization via flux projection

Define the attractive operator L_{att} by its Fokker–Planck action on p through the flux $J^{\text{att}}(p) = p\, v^{\text{att}} - D\nabla p$, where v^{att} is the projection (5.2) evaluated at p. The residual operator is then induced by the residual flux $J^{\text{res}}(p) = J(p) - J^{\text{att}}(p)$. With this construction,

$$\big\langle \nabla r(p),\, L_{\text{att}}p \big\rangle \;=\; -\int \langle \nabla\phi,\, p\, v^{\text{att}}\rangle\, dx\, dy \;=\; -\frac{\langle v, \nabla\phi\rangle^2_{L^2(p)}}{\|\nabla\phi\|^2_{L^2(p)}} \;\le\; 0, \tag{5.3}$$

and $\langle\nabla r(p), L_{\text{res}}p\rangle = 0$ by orthogonality, which verifies the contraction and neutrality conditions [17].

5.3 Uniqueness under neutrality and boundary conditions

The flux projection (5.2) is unique given p and the choice of inner product $\langle\cdot,\cdot\rangle_{L^2(p)}$. For domains with reflecting boundaries or fast decay at infinity, the weighted Hodge decomposition (5.1) implies that any alternative attractive choice that achieves the same maximal MI gain must coincide with (5.2) almost everywhere. Consequently, the induced operator splitting is unique up to null sets and gradient potentials differing by constants [6].

5.4 Neutral transports and symmetry classes

Neutral transports can be characterized by the kernel of the functional $v \mapsto \langle v, \nabla\phi\rangle_{L^2(p)}$. Typical representatives include: (i) divergence-free flows with respect to p (the $v^\perp$ in (5.1)); (ii) isomarginal flows that preserve p^X and p^Y; (iii) symmetry-induced motions along group orbits that leave ϕ invariant. Each of these classes satisfies $\langle\nabla r(p), L_{\text{res}}p\rangle = 0$ to first order, and under the standing assumptions will not increase r [18].

5.5 Discrete and graph-based settings

On a finite product space $\mathcal{X}\times\mathcal{Y}$ with a reversible Markov kernel, the flux reads $J(e) = p(i)\,K(i,j) - p(j)\,K(j,i)$ on edges $e = (i \to j)$ and admits a discrete Hodge decomposition into gradient and cycle flows. The MI potential becomes $\phi(i,j) = \log\frac{p(i,j)}{p^X(i)p^Y(j)}$, and the projection (5.2) is implemented with the counting measure weighted by p; all statements above carry over mutatis mutandis, with orthogonality taken in $\ell^2(p)$ [8].

5.6 Summary

The velocity-field viewpoint yields a canonical and verifiable realization of the operator splitting $L = L_{\text{att}} + L_{\text{res}}$. The attractive part is the $L^2(p)$ projection of the velocity onto the MI-gradient direction, ensuring strict contraction of r, whereas the residual part is orthogonal to $\nabla\phi$ and hence neutral. Uniqueness follows from weighted Hodge theory under standard boundary conditions, and the construction extends to discrete settings. These ingredients underpin model-specific derivations and estimates in the subsequent sections [17, 18].

6 Applications and Model-Specific Consequences

This section illustrates how the operator-level characterization of natural forces yields concrete, verifiable predictions in representative models. We emphasize criteria that can be checked directly from the generator and flux representations developed in Sections 3–5.

6.1 Gaussian–linear models: exact criteria

Consider the coupled Ornstein–Uhlenbeck setting of Section 3.3 with state $Z_t = (X_t, Y_t) \in \mathbb{R}^{d_x+d_y}$ and dynamics

$$dZ_t = AZ_t\,dt + \sqrt{2\Sigma}\,dW_t, \qquad A = \begin{bmatrix} -\alpha I & B \\ C & -\gamma I \end{bmatrix}, \quad \Sigma \succ 0, \tag{6.1}$$

where $B \in \mathbb{R}^{d_x \times d_y}$ and $C \in \mathbb{R}^{d_y \times d_x}$. Let $\Gamma_t = \mathrm{Cov}(Z_t)$ solve the Lyapunov equation $\dot{\Gamma}_t = A\Gamma_t + \Gamma_t A^\top + 2\Sigma$ [6]. Writing $\Gamma_t = \begin{bmatrix} \Gamma_{xx}(t) & \Gamma_{xy}(t) \\ \Gamma_{yx}(t) & \Gamma_{yy}(t) \end{bmatrix}$, one has for Gaussian states that

$$\mathrm{MI}(X_t; Y_t) \;=\; \tfrac{1}{2} \log \frac{\det \Gamma_{xx}(t)\, \det \Gamma_{yy}(t)}{\det \Gamma_t}. \tag{6.2}$$

Differentiating (6.2) and using $\dot{\Gamma}_t$ yields an explicit expression for $\frac{d}{dt} r(t)$; in particular,

$$\frac{d}{dt} r(t) \;\le\; 0 \qquad \iff \qquad \mathsf{Sym}\big(B\Gamma_{yx}(t) + C\Gamma_{xy}(t)\big) \;\succeq\; 0, \tag{6.3}$$

under coercivity of the symmetric part $\mathsf{Sym}(A)$ and $\Sigma \succ 0$. Hence the cross-drift block controls r-contraction through the instantaneous cross-covariance [6].

6.2 Nonlinear drift–diffusion with MI forcing

For a Fokker–Planck generator with drift $b = (b_x, b_y)$ and diffusion tensor D,

$$Lp = -\nabla \cdot (p\, b) + \nabla \cdot (D \nabla p), \tag{6.4}$$

introduce the MI potential $\phi = \log \frac{p}{p^X p^Y}$ and the projected attractive drift

$$b_{\mathrm{att}} \;=\; \frac{\langle b, \nabla\phi \rangle_{L^2(p)}}{\|\nabla\phi\|^2_{L^2(p)}} \nabla\phi, \qquad b_{\mathrm{res}} \;=\; b - b_{\mathrm{att}}. \tag{6.5}$$

Then the generator splitting induced by $b = b_{\mathrm{att}} + b_{\mathrm{res}}$ satisfies the contraction/neutrality conditions from Section 3. Moreover, if b_{res} is p-divergence free, i.e. $\nabla \cdot (p\, b_{\mathrm{res}}) = 0$, then L_{res} is neutral and (6.5) yields a verifiable recipe for computing L_{att} from data-driven estimates of p and ϕ [5, 18].

6.3 Discrete-state systems and Markov chains

Let P be a transition kernel on $\mathcal{X} \times \mathcal{Y}$ with generator $L = P - I$. Writing the flux on edges $e = (i \to j)$ as $J(e) = p(i)\, P(i,j) - p(j)\, P(j,i)$ and the discrete MI potential as $\phi(i) = \log \frac{p(i)}{p^X(i_x) p^Y(i_y)}$, define

$$J^{\mathrm{att}}(e) \;=\; \frac{\sum_{e'} J(e')\, (\phi(j') - \phi(i'))}{\sum_{e'} (\phi(j') - \phi(i'))^2}\, (\phi(j) - \phi(i)), \qquad J^{\mathrm{res}}(e) = J(e) - J^{\mathrm{att}}(e). \tag{6.6}$$

Then J^{att} maximizes the instantaneous MI gain over rank-one gradient edge flows and induces a discrete attractive generator L_{att} that contracts r to first order; J^{res} is neutral in the $\ell^2(p)$ pairing [8].

6.4 Robustness to estimation error

Suppose $\widehat{p}$ and $\widehat{\phi}$ are plug-in estimates of p and ϕ, and denote the corresponding attractive operator by $\widehat{L}_{\mathrm{att}}$. If $\|\widehat{\nabla\phi} - \nabla\phi\|_{L^2(p)} \le \varepsilon$ and $\|\widehat{v} - v\|_{L^2(p)} \le \varepsilon$ for the underlying velocity field, then the contraction defect satisfies

$$\Big| \big\langle \nabla r(p),\, (\widehat{L}_{\mathrm{att}} - L_{\mathrm{att}}) p \big\rangle \Big| \;\lesssim\; \varepsilon \left(\|v\|_{L^2(p)} + \|\nabla\phi\|_{L^2(p)} \right), \tag{6.7}$$

and remains dominated by the nominal dissipation rate when ε is sufficiently small; cf. the perturbation stability in Theorem 4.4.

6.5 Guidelines for empirical verification

Given a time series or ensemble of snapshots, the following steps provide a pragmatic pipeline:

1. Estimate p, p^X, p^Y, and $\phi = \log \frac{p}{p^X p^Y}$ (e.g., via kernel density or score models) [5].
2. Compute velocities (or fluxes) and project onto $\nabla\phi$ as in (6.5) or (6.6) to obtain L_{att}.
3. Evaluate $\frac{d}{dt}r$ empirically using the variational pairing with L_{att} (and neutrality checks for L_{res}).
4. Quantify robustness using bounds of the form (6.7) and cross-validate with synthetic data generated from the fitted dynamics.

Summary. The operator characterization leads to tractable, testable criteria across Gaussian, nonlinear diffusive, and discrete models. It enables data-driven construction of the attractive generator and provides robustness guarantees under estimation error, thereby connecting the abstract theory to empirical practice [5, 6, 8, 17, 18].

7 Discussion and Outlook

We have proposed an operator-theoretic formulation of natural forces in which the force-carrying content of an interaction is identified with the component of the generator that provably contracts the information radius r and, equivalently, generates mutual-information (MI) gradient flow. This viewpoint unifies geometric, information-theoretic, and semigroup-based descriptions and yields verifiable criteria at the level of fluxes, drifts, and discrete transitions.

Conceptual implications. By elevating forces from fields on configuration space to components of generators on spaces of probability measures, interactions are characterized in a coordinate-free manner through their informational effects. The attractive component L_{att} emerges as the mechanism that creates or preserves informational coupling, while L_{res} encodes neutral transports, symmetries, and marginal-preserving operations. This distinction clarifies when a given evolution is *force-like* in the informational sense and connects to classical notions via Fokker–Planck and Langevin representations [6].

Methodological consequences. The flux-projection construction (Section 5) provides a canonical recipe for isolating L_{att} from data or models through a single variational step onto the MI-gradient direction, immediately implying a dissipation inequality for r. For Gaussian–linear systems (Section 6.1), this reduces to spectral conditions on cross-drift blocks; for nonlinear diffusions (Section 6.2), it yields a constructive decomposition of the drift; and for discrete chains (Section 6.3), it becomes a projection of edge fluxes in $\ell^2(p)$ [8, 5].

Limitations. Several caveats merit emphasis. First, the equivalence with MI-gradient flows requires specifying a metric g_p on the statistical manifold and ensuring sufficient regularity for variational derivatives (Section 2.3); different choices of g_p may lead to distinct steepest-ascent dynamics [5]. Second, neutrality of L_{res} can be ensured in multiple, model-dependent ways (marginal preservation, anti-self-adjointness, symmetry), and uniqueness may require boundary and decay conditions (Section 5.3). Third, the MI potential $\phi = \log \frac{p}{p^X p^Y}$ is nonlocal through its marginals; well-posedness demands positivity and smoothness that may be challenging in high dimensions [6].

Relations to prior work. Our results synthesize variational identities for entropy-like functionals, semigroup calculus, and information-geometric perspectives [4, 5, 6]. The contraction of r along MI-aligned flows extends previously established results (L3/L4) to an operator level and provides neutrality criteria compatible with velocity-field decompositions. In this sense, the present framework can be viewed as a generator-based completion of earlier formulations [17, 18].

Open directions. We highlight several directions for future work:

1. **Repulsive informational effects.** Extending the present attractive theory to repulsive components L_{rep} that provably *increase* r (or reduce MI) while remaining consistent with conservation constraints and thermodynamic bounds.

2. **Non-Markovian dynamics.** Generalizing to memory-dependent evolutions (e.g., fractional generators or path-dependent drifts) where generator calculus is replaced by suitable history operators, and clarifying which parts retain informational contraction.

3. **Interacting fields and gauge symmetries.** Formulating neutrality through group actions that leave ϕ invariant and analyzing how broken symmetries induce attractive directions; relating these notions to geometric dissipation structures.

4. **Numerical schemes.** Designing structure-preserving discretizations that implement the flux projection at the PDE or Markov-chain level and inherit discrete dissipation inequalities (cf. Section 4.4).

5. **Data-driven estimation.** Developing statistically stable estimators of L_{att} from finite samples, with rates controlled by MI-gradient signal-to-noise and robustness guarantees as in (6.7).

Concluding remarks. The operator characterization presented here provides a compact and robust lens on natural forces as generators of informational coupling. Besides its theoretical economy, the framework offers practical diagnostics and constructive algorithms for isolating force-carrying components in complex stochastic systems. We expect that further developments along the aforementioned directions will broaden its applicability across physics, information theory, and data-driven modeling [4, 8, 5, 6].

Appendix A. Variational Identities and Contraction Lemmas

This appendix collects proofs of the variational identities and basic contraction results used in Sections 2–4. Throughout, we work under the standing assumptions of Section 2.5 and use the same notation.

A.1. Proof of the generator variational identity

We first justify the chain-rule formula (2.6) for sufficiently regular functionals $\mathcal{F}[p]$ on densities.

Lemma .1. *Let $\mathcal{F}$ be a Fréchet differentiable functional on a suitable Banach space of densities containing p_t, with derivative represented by $\delta\mathcal{F}/\delta p \in L^2$ in the sense that*

$$\frac{d}{d\varepsilon}\,\mathcal{F}[p+\varepsilon h]\Big|_{\varepsilon=0} \;=\; \big\langle \tfrac{\delta \mathcal{F}}{\delta p}(p),\, h\big\rangle \qquad \text{for all admissible perturbations } h. \tag{.1}$$

If p_t solves $\partial_t p_t = L p_t$ with generator L and $t \mapsto p_t$ is differentiable in the ambient space, then

$$\frac{d}{dt}\,\mathcal{F}[p_t] \;=\; \Big\langle \frac{\delta \mathcal{F}}{\delta p}(p_t),\, \partial_t p_t\Big\rangle \;=\; \Big\langle \frac{\delta \mathcal{F}}{\delta p}(p_t),\, L p_t\Big\rangle \;=\; \big\langle L^* \tfrac{\delta \mathcal{F}}{\delta p}(p_t),\, p_t\big\rangle, \tag{.2}$$

where L^* is the L^2-adjoint of L.

Proof. By differentiability of $t \mapsto p_t$ and the chain rule for Fréchet derivatives,

$$\frac{d}{dt}\,\mathcal{F}[p_t] \;=\; \Big\langle \frac{\delta \mathcal{F}}{\delta p}(p_t),\, \partial_t p_t\Big\rangle. \tag{.3}$$

Substituting $\partial_t p_t = L p_t$ yields the second equality. If L is densely defined and admits an L^2-adjoint L^* on a core containing $\delta\mathcal{F}/\delta p(p_t)$, an integration-by-parts argument (or semigroup duality) gives

$$\Big\langle \frac{\delta \mathcal{F}}{\delta p}(p_t),\, L p_t\Big\rangle \;=\; \big\langle L^* \tfrac{\delta \mathcal{F}}{\delta p}(p_t),\, p_t\big\rangle. \tag{.4}$$

This yields the desired identity. □

A.2. Proof of the mutual-information identity

We now specialize Lemma .1 to $\mathcal{F} = \mathrm{MI}(X;Y)$. Recall

$$\mathrm{MI}(X;Y) \;=\; \int_{\mathcal{X}\times\mathcal{Y}} p(x,y)\,\log\frac{p(x,y)}{p^X(x)\,p^Y(y)}\,dx\,dy, \tag{.5}$$

where p^X and p^Y denote the marginals. Under the positivity and integrability assumptions of Section 2.5, we may differentiate under the integral sign to obtain the variational derivative.

Lemma .2. *Under the standing assumptions, the variational derivative of* MI *with respect to p is*

$$\frac{\delta\,\mathrm{MI}}{\delta p}(p) \;=\; \log\frac{p}{p^X p^Y}, \tag{.6}$$

and the time derivative along $\partial_t p_t = L p_t$ is given by

$$\frac{d}{dt}\,\mathrm{MI}(X_t;Y_t) \;=\; \int \log\frac{p_t(x,y)}{p_t^X(x)p_t^Y(y)}\,\big(L p_t\big)(x,y)\,dx\,dy. \tag{.7}$$

Proof. For a perturbation h with zero total mass, the directional derivative is

$$\frac{d}{d\varepsilon}\,\mathrm{MI}[p+\varepsilon h]\Big|_{\varepsilon=0} = \int h\,\log\frac{p}{p^X p^Y}\,dx\,dy \;+\; \int p\,\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\log\frac{p+\varepsilon h}{(p^X+\varepsilon h^X)(p^Y+\varepsilon h^Y)}\,dx\,dy, \tag{.8}$$

where $h^X(x) = \int h(x,y)\,dy$ and $h^Y(y) = \int h(x,y)\,dx$. The second term cancels by normalization of marginals, leaving $\int h\,\log\frac{p}{p^X p^Y}\,dx\,dy$, which identifies $\delta\mathrm{MI}/\delta p$ as claimed. Applying Lemma .1 with $\mathcal{F} = \mathrm{MI}$ yields the time derivative along Lp. □

A.3. Proof of Lemma 2.1

We recall Lemma 2.1: if $\langle \nabla r(t), L_{\mathrm{att}} p_t\rangle \le 0$ for all $t \in [0,T]$ and L_{res} is neutral, then $r(t)$ is nonincreasing, with strict decrease when the inequality is strict on a set of positive measure.

Proof of Lemma 2.1. Combining the variational identity (2.10) with neutrality of L_{res} gives

$$\frac{d}{dt}\,r(t) \;=\; \big\langle \nabla r(t),\, L_{\mathrm{att}} p_t \big\rangle \;\le\; 0 \qquad \text{for all } t \in [0,T]. \tag{.9}$$

Integration over $[s,t] \subset [0,T]$ yields

$$r(t) - r(s) \;=\; \int_s^t \frac{d}{d\tau}\,r(\tau)\,d\tau \;\le\; 0, \tag{.10}$$

so $r(t)$ is nonincreasing. If the integrand is strictly negative on a measurable subset $E \subset [0,T]$ of positive measure, then the integral over any interval intersecting E is strictly negative, implying a strict decrease of $r(t)$ on that interval. □

A.4. Proof of Theorem 4.1 (sketch)

We provide additional details for Theorem 4.1, which states the equivalence between operator contraction, MI-gradient generation, and a dissipation inequality of EVI type.

Proof sketch of Theorem 4.1. (i)⇒(ii): Under the metric g_{p_t}, variational derivatives admit a Riesz representation, so there exists a tangent vector G_t such that

$$\big\langle \nabla r(t),\, \xi \big\rangle \;=\; g_{p_t}(G_t, \xi) \qquad \text{for all tangent directions } \xi. \tag{.11}$$

The identity $\frac{d}{dt}r(t) = \langle \nabla r(t), L_{\mathrm{att}} p_t\rangle$ then reads $g_{p_t}(G_t, L_{\mathrm{att}} p_t) \le 0$. By definition of gradient flow, the steepest descent direction of r (and steepest ascent of MI) is proportional to G_t, so $L_{\mathrm{att}} p_t$ coincides with the gradient-flow direction up to a scalar factor; reparametrization of time yields the corresponding flow [5].

(ii)⇒(iii): Standard energy–dissipation arguments for gradient flows in metric spaces (here with metric induced by g_p) yield an inequality of the form

$$r(t) - r(s) \;\le\; -\int_s^t \big\|\nabla r(\tau)\big\|^2_{g_{p_\tau}}\,d\tau \;+\; \kappa \int_s^t \Psi(p_\tau)\,d\tau, \tag{.12}$$

where Ψ captures non-gradient contributions [6]. In our setting, these arise precisely from L_{res}, giving (4.1). If L_{res} is neutral, then $\Psi \equiv 0$ and one may take $\kappa = 0$.

(iii)⇒(i): Differentiating the EVI-type inequality (4.1) with respect to t at $t = s$ recovers $\frac{d}{dt}r(t) \le -\|\nabla r(t)\|^2_{g_{p_t}} + \kappa\Psi(p_t)$. Neutrality of L_{res} implies $\Psi \equiv 0$, so the right-hand side is nonpositive and coincides with $\langle \nabla r(t), L_{\mathrm{att}} p_t\rangle$, giving (i). This completes the sketch. □

A.5. Remarks on metrics and domains

For completeness, we record two technical remarks.

Remark .3 (Choice of metric). The metric g_p may be taken as the Fisher–Rao metric, a Wasserstein-type metric, or another information-geometric metric under which r is sufficiently regular [5]. The precise form of the gradient-flow equations depends on this choice, but the operator-level statements formulated in terms of variational pairings remain valid as long as the Riesz representation holds.

Remark .4 (Domains of generators). In all arguments above, it is implicitly assumed that the densities p_t remain in the domain of L (and L^*) and that the variational derivatives $\delta\mathcal{F}/\delta p(p_t)$ belong to the domain of L^* on a common core [8, 6]. These conditions can be verified on a case-by-case basis for the models considered in Section 6.

Appendix B. Neutrality, Symmetry, and Flux Decompositions

This appendix provides details for the neutrality results in Section 4 and the flux-based constructions of Section 5. We retain the notation and assumptions of Sections 2–5.

B.1. Proof of Proposition 4.2

We recall Proposition 4.2: if L_{res} preserves marginals to first order and satisfies the commutation condition

$$\big\langle \log(p_t^X) + \log(p_t^Y),\, L_{\mathrm{res}} p_t \big\rangle = 0, \tag{.1}$$

then L_{res} is neutral with respect to r.

Proof of Proposition 4.2. By definition,

$$r(t) = H(X_t) + H(Y_t) - 2\,\mathrm{MI}(X_t; Y_t). \tag{.2}$$

Differentiating along L_{res} and using Lemma .1 yields

$$\frac{d}{d\varepsilon} r(p_t + \varepsilon L_{\mathrm{res}} p_t)\Big|_{\varepsilon=0} = \big\langle \nabla r(t),\, L_{\mathrm{res}} p_t \big\rangle = \frac{d}{dt} r(t)\Big|_{\partial_t p_t = L_{\mathrm{res}} p_t}. \tag{.3}$$

We compute this derivative by parts. For the marginal entropies,

$$\begin{aligned} \frac{d}{dt} H(X_t)\Big|_{L_{\mathrm{res}}} &= -\int \big(1 + \log p_t^X(x)\big)\, \partial_t p_t^X(x)\, dx \\ &= -\int \big(1 + \log p_t^X(x)\big)\, \Pi_X L_{\mathrm{res}} p_t(x)\, dx, \end{aligned} \tag{.4}$$

$$\begin{aligned} \frac{d}{dt} H(Y_t)\Big|_{L_{\mathrm{res}}} &= -\int \big(1 + \log p_t^Y(y)\big)\, \partial_t p_t^Y(y)\, dy \\ &= -\int \big(1 + \log p_t^Y(y)\big)\, \Pi_Y L_{\mathrm{res}} p_t(y)\, dy. \end{aligned} \tag{.5}$$

By marginal preservation, $\Pi_X L_{\mathrm{res}} p_t = 0$ and $\Pi_Y L_{\mathrm{res}} p_t = 0$, so both derivatives vanish:

$$\frac{d}{dt} H(X_t)\Big|_{L_{\mathrm{res}}} = \frac{d}{dt} H(Y_t)\Big|_{L_{\mathrm{res}}} = 0. \tag{.6}$$

For the mutual information, we use Lemma .2 to obtain

$$\frac{d}{dt}\,\mathrm{MI}(X_t; Y_t)\Big|_{L_{\mathrm{res}}} = \int \log \frac{p_t(x,y)}{p_t^X(x) p_t^Y(y)}\, \big(L_{\mathrm{res}} p_t\big)(x,y)\, dx\, dy. \tag{.7}$$

Decomposing the logarithm and rearranging gives

$$\begin{aligned} \int \log \frac{p_t}{p_t^X p_t^Y}\, L_{\mathrm{res}} p_t &= \int \log p_t\, L_{\mathrm{res}} p_t - \int \log p_t^X(x)\, L_{\mathrm{res}} p_t(x,y)\, dx\, dy - \int \log p_t^Y(y)\, L_{\mathrm{res}} p_t(x,y)\, dx\, dy \\ &=: I_1 - I_2 - I_3. \end{aligned} \tag{.8}$$

By mass conservation of L_{res} (implicit in the semigroup setting), the contribution of the constant part of $\log p_t$ in I_1 vanishes. The terms I_2 and I_3 can be rewritten using the marginalization operators:

$$I_2 = \int \log p_t^X(x)\, \Pi_X L_{\mathrm{res}} p_t(x)\, dx, \tag{.9}$$

$$I_3 = \int \log p_t^Y(y)\, \Pi_Y L_{\mathrm{res}} p_t(y)\, dy, \tag{.10}$$

which vanish by marginal preservation. The remaining contribution reduces, by the assumed commutation condition, to

$$I_1 = \big\langle \log p_t^X + \log p_t^Y\,,\, L_{\text{res}} p_t \big\rangle = 0. \tag{.11}$$

Thus $\frac{d}{dt}\text{MI}(X_t; Y_t)|_{L_{\text{res}}} = 0$, and hence

$$\big\langle \nabla r(t),\, L_{\text{res}} p_t \big\rangle = \frac{d}{dt} r(t)\Big|_{L_{\text{res}}} = 0. \tag{.12}$$

This proves neutrality. □

B.2. Proof of Proposition 4.3

We next recall Proposition 4.3: if L_{res} is anti-self-adjoint with respect to the pairing induced by g_p on the subspace tangent to isomarginal deformations and preserves the product structure $p^X p^Y$, then L_{res} is neutral.

Proof of Proposition 4.3. Let $\mathcal{M}$ denote the manifold of densities with fixed marginals p^X and p^Y, and let $T_p\mathcal{M}$ be its tangent space at p. The metric g_p induces an inner product on $T_p\mathcal{M}$. By assumption, $L_{\text{res}} p \in T_p\mathcal{M}$ and

$$g_p\big(L_{\text{res}} p,\, \xi\big) \;=\; -g_p\big(\xi,\, L_{\text{res}} p\big) \qquad \text{for all } \xi \in T_p\mathcal{M}. \tag{.13}$$

On the other hand, the gradient of r restricted to $\mathcal{M}$ is a tangent vector $\text{grad}_{\mathcal{M}} r(p) \in T_p\mathcal{M}$ satisfying

$$\big\langle \nabla r(p),\, \xi \big\rangle \;=\; g_p\big(\text{grad}_{\mathcal{M}} r(p),\, \xi\big) \qquad \text{for all } \xi \in T_p\mathcal{M}, \tag{.14}$$

by the Riesz representation associated with g_p. In particular,

$$\big\langle \nabla r(p),\, L_{\text{res}} p \big\rangle \;=\; g_p\big(\text{grad}_{\mathcal{M}} r(p),\, L_{\text{res}} p\big). \tag{.15}$$

Anti-self-adjointness with respect to g_p on $T_p\mathcal{M}$ implies that

$$g_p\big(\text{grad}_{\mathcal{M}} r(p),\, L_{\text{res}} p\big) \;=\; -g_p\big(L_{\text{res}} p,\, \text{grad}_{\mathcal{M}} r(p)\big) \;=\; -g_p\big(\text{grad}_{\mathcal{M}} r(p),\, L_{\text{res}} p\big), \tag{.16}$$

hence the quantity must vanish. Thus

$$\big\langle \nabla r(p),\, L_{\text{res}} p \big\rangle = 0, \tag{.17}$$

which is precisely neutrality. Preservation of $p^X p^Y$ ensures that L_{res} indeed maps into $T_p\mathcal{M}$ and does not spuriously generate mutual information. □

B.3. Details on the flux projection

We provide a derivation of the directional dissipation formula (5.3). Recall that for a drift–diffusion model with flux $J = pv - D\nabla p$ and MI potential $\phi = \log \frac{p}{p^X p^Y}$,

$$v^{\text{att}} = \frac{\langle v, \nabla\phi \rangle_{L^2(p)}}{\|\nabla\phi\|^2_{L^2(p)}} \nabla\phi, \qquad v^{\text{res}} = v - v^{\text{att}}. \tag{.18}$$

Lemma .5. *With L_{att} induced by the flux $J^{\text{att}} = pv^{\text{att}} - D\nabla p$, one has*

$$\big\langle \nabla r(p),\, L_{\text{att}} p \big\rangle = -\frac{\langle v, \nabla\phi \rangle^2_{L^2(p)}}{\|\nabla\phi\|^2_{L^2(p)}} \;\le\; 0. \tag{.19}$$

Proof. Using Lemma .2, one can show that the first variation of r along a perturbation δp can be written in terms of ϕ and the associated flux. For drifts of the form $L^{(v)}p = -\nabla\cdot(pv)$, a standard computation gives

$$\left\langle \nabla r(p),\, L^{(v)}p \right\rangle = -\int \langle \nabla\phi,\, pv\rangle \, dx\, dy. \tag{.20}$$

(Here the diffusion part contributes a nonpositive term independent of the decomposition of v and can be absorbed into L_{res} for the purpose of directional analysis.) Substituting v^{att} yields

$$\begin{aligned}\left\langle \nabla r(p),\, L_{\mathrm{att}}p \right\rangle &= -\int \left\langle \nabla\phi,\, p\,\frac{\langle v, \nabla\phi\rangle_{L^2(p)}}{\|\nabla\phi\|^2_{L^2(p)}}\,\nabla\phi \right\rangle dx\, dy \\ &= -\frac{\langle v, \nabla\phi\rangle_{L^2(p)}}{\|\nabla\phi\|^2_{L^2(p)}} \int p\,\|\nabla\phi\|^2 \, dx\, dy \\ &= -\frac{\langle v, \nabla\phi\rangle^2_{L^2(p)}}{\|\nabla\phi\|^2_{L^2(p)}},\end{aligned} \tag{.21}$$

which is nonpositive. This proves the claim. □

B.4. Discrete Hodge decomposition

Finally, we briefly comment on the discrete analogue used in Section 5.5. On a finite graph with vertex set V and edge set E, endowed with a reversible Markov kernel K, any flux $J : E \to \mathbb{R}$ with zero net flow at each vertex admits a decomposition

$$J = J^{\mathrm{grad}} + J^{\mathrm{cycle}}, \tag{.22}$$

where J^{grad} is a gradient flow of a potential on V and J^{cycle} lies in the cycle space of the graph. With the weighted inner product $\langle J_1, J_2\rangle_{\ell^2(p)} = \sum_{e\in E} J_1(e)J_2(e)/w(e)$ for suitable edge weights $w(e)$ (e.g. proportional to p and K), this decomposition is orthogonal [8]. The projection (6.6) is precisely the orthogonal projection of J onto the one-dimensional subspace spanned by the discrete gradient of the MI potential, and the neutrality of the residual follows exactly as in the continuous case by orthogonality to the discrete gradient.

These results complete the proofs and structural statements underlying the neutrality and flux-decomposition arguments in the main text.

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