# New theory to find the global optimum for nonconvex optimization problems

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***Abstract*—This paper proposes a new theory for calculating the global optimum of nonconvex optimization problem. Key idea used in this paper is a combination of probability-one homotopy method and von Neumann's minimax theorem. Brouwer's fixed point theorem is used for the connection of these two theories.**

***Index Terms*—nonconvex optimization, global optimum, probability-one homotopy method, minimax theorem, fixed point theorem**

## I. INTRODUCTION

Nonconvex optimization is still recognized as difficult problem to solve, while satisfactory results are obtained for convex optimization theory, such as linear programming [1]- [3] and linear matrix inequality [4], [5] used in automatic control theory. Effective tool for general nonconvex optimization problem may be only so-called Karush-Kuhn-Tucker (KKT) condition [6], [7]. However, KKT condition is a necessary, not necessary and sufficient, condition and may not be easy to solve in general.

Of course, there are many studies in the field of nonconvex optimization. A detailed survey in this field of research is shown in e.g. [17]. As shown in [17], some studies use hidden convexity or derive analytic solution to nonconvex problems (e.g. [18], [19]). Also, many studies are devoted to various types of gradient methods, which may converge to local minimum ( [20] etc.).

The purpose of the present paper is to propose a new approach for nonconvex optimization problem. For this purpose, we use, as our key tool, a combination of homotopy method and game theory, or more specifically a combination of probability-one homotopy method [8] and von Neumann's minimax theorem [11].

In the field of probability-one homotopy method, many results are obtained ( [9], [10]etc). To the present author's knowledge, the origin of probability-one homotopy method may be [8]. In [8], a homotopy that converges from arbitrary initial point to a fixed point of given continuous and bounded mapping is proposed. By using such homotopy we can calculate a fixed point.

On the other hand, von Neumann's minimax theorem [11] clarifies that, if a point called saddle point exists for some function, that point is an optimal solution to a minimax problem corresponding to that function. It is explained in e.g. [13] that a saddle point of minimax problem shown by von Neumann in [11] etc. can be recognized as a fixed point.

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Therefore, the probability-one homotopy method shown in [8] and von Neumann's minimax theorem [11] are connected by the concept of fixed point.

Combining these two theories, the present author is led to an idea that if nonconvex optimization problem can be equivalently transformed to a minimax problem whose saddle point always exists then we can calculate this saddle point (i.e. a solution to the nonconvex optimization problem) by using appropriate homotopy. The present paper shows that this idea is successful in case that given nonconvex problem is polynomial-type one. (The definition of "polynomial-type" is given later.)

As is well known, one of the most important nonconvex problem in the field of control theory is bilinear matrix inequality (BMI). In this note, we consider general nonconvex problem as pre-stage of BMI.

In the next section, we start our argument by equivalently transforming nonconvex optimization problem to the set of quadratic inequalities. In Section 3, it is further transformed to a two-person zero-sum game (or minimax problem). In Section 4, a continuous mapping is introduced and Brouwer's fixed point theorem is applied to it. By using that result, homotopy that converges to the saddle point is proposed in Section 5. A numerical example is studied in Section 6.

## II. TRANSFORMATION OF NONCONVEX PROBLEM TO SQI

We consider a nonconvex optimization problem given as

$$f_1^{NC}(z_1, \ldots, z_{p1}) \to min \tag{1}$$

subject to inequality constraints

$$\begin{cases} f_2^{NC}(z_1, \ldots, z_{p1}) \leq 0 \\ \quad\vdots \\ f_{q1}^{NC}(z_1, \ldots, z_{p1}) \leq 0 \end{cases} \tag{2}$$

and equality constraints

$$\begin{cases} f_{q1+1}^{NC}(z_1, \ldots, z_{p1}) = 0 \\ \quad\vdots \\ f_{q2}^{NC}(z_1, \ldots, z_{p1}) = 0 \end{cases} \tag{3}$$

where $q_1 \geq 2$ and $q_2 \geq q_1 + 1$.

This is general formulation of nonconvex problem given in e.g. [15]. Throughout this note, $f_1^{NC}, \ldots, f_{q_2}^{NC}$ are restricted to real-polynomials in $z_1, \ldots, z_{p_1}$. Such problem is called polynomial-type nonconvex optimization problem in this note.

In the followings, we equivalently transform (1)-(3) to the set of quadratic inequalities (SQI). SQI is defined as the set of inequalities of the form

$$\left\{\begin{array}{l} f_1(z_1,\ldots,z_p)\le 0 \\ \quad\vdots \\ f_q(z_1,\ldots,z_p)\le 0 \end{array}\right. \tag{4}$$

where $f_i$ are all real-polynomials in $z_1,\ldots,z_p$ and the order of all $f_i$ is less than or equal to 2, i.e. $f_i$ is given by

$$f_i(z)=z^TA_iz+2b_i^Tz+c_i \tag{5}$$

for some constant $A_i,b_i,c_i$ and $z:=[z_1,\ldots,z_p]^T$.

First, we use so-called $\gamma$-iterations. Namely (1) is replaced by

$$f_1^{NC}(z_1,\ldots,z_{p1})-\gamma\le 0 \tag{6}$$

where $\gamma$ is iteratively prescribed constant.

Next, we make the order of all left-hand sides of (2),(3) and (6) lower than or equal to 2 by introducing new unknowns. By this, we obtain

$$\left\{\begin{array}{l} f_1^Q(z_1,\ldots,z_{p2})-\gamma\le 0 \\ f_2^Q(z_1,\ldots,z_{p2})\le 0 \\ \quad\vdots \\ f_{q1}^Q(z_1,\ldots,z_{p2})\le 0 \\ f_{q1+1}^Q(z_1,\ldots,z_{p2})=0 \\ \quad\vdots \\ f_{q3}^Q(z_1,\ldots,z_{p2})=0 \end{array}\right. \tag{7}$$

where $p_2\ge p_1$, $q_3\ge q_2$ and the order of all $f_i^Q$'s are less than or equal to 2. $z_{p1+1},\ldots,z_{p2}$ are new unknowns. This transformation is always possible. See Appendix I.

The equations in (7) can be equivalently transformed to inequalities by using an obvious neccessary and sufficient relationship given by

$$\alpha=0\Leftrightarrow\alpha\le 0\text{and}-\alpha\le 0 \tag{8}$$

By using this, (7) is equivalent to

$$\left\{\begin{array}{l} f_1^Q(z_1,\ldots,z_{p2})-\gamma\le 0 \\ f_2^Q(z_1,\ldots,z_{p2})\le 0 \\ \quad\vdots \\ f_{q1}^Q(z_1,\ldots,z_{p2})\le 0 \\ f_{q1+1}^Q(z_1,\ldots,z_{p2})\le 0 \\ -f_{q1+1}^Q(z_1,\ldots,z_{p2})\le 0 \\ \quad\vdots \\ f_{q3}^Q(z_1,\ldots,z_{p2})\le 0 \\ -f_{q3}^Q(z_1,\ldots,z_{p2})\le 0 \end{array}\right. \tag{9}$$

This is a SQI.

## III. SQI AS GAME

In this section, we consider SQI from the viewpoint of game theory. Namely, we equivalently transform a general SQI given by (4) to a minimax problem.

For this purpose, first transform unknowns $z_1,\ldots,z_p$ to new unknowns $x_1,\ldots,x_{p+1}$ by

$$\left\{\begin{array}{l} z_1=\frac{x_1}{x_{p+1}} \\ \quad\vdots \\ z_p=\frac{x_p}{x_{p+1}} \end{array}\right. \tag{10}$$

where

$$x_{p+1}\ne 0 \tag{11}$$

must be satisfied.

Define

$$\hat f_i(x_1,\ldots,x_{p+1}):=x_{p+1}^2f_i(\frac{x_1}{x_{p+1}},\ldots,\frac{x_p}{x_{p+1}})\qquad (i=1,\ldots,q)$$

Then (4) is equivalent to a set of quadratic homogeneous inequalities given by

$$\left\{\begin{array}{l} \hat f_1(x_1,\ldots,x_{p+1})\le 0 \\ \quad\vdots \\ \hat f_q(x_1,\ldots,x_{p+1})\le 0 \end{array}\right. \tag{12}$$

Obviously if $f_i$ is given by (5) then $\hat f_i$ is given by

$$\hat f_i(x)=x^T\begin{bmatrix} A_i & b_i \\ b_i^T & c_i\end{bmatrix}x$$

where $x:=[x_1,\ldots,x_{p+1}]^T$.

Furthermore (11) is approximately equivalent to

$$\hat f_{q+1}(x_1,\ldots,x_{p+1}):=-x_{p+1}^2+\epsilon\le 0 \tag{13}$$

where $\epsilon$ is sufficiently small positive constant.

If $(x_1,\ldots,x_{p+1})$ is a solution to (10) for given $z_i$ then $(\alpha x_1,\ldots,\alpha x_{p+1})$ is also solution for any scalar $\alpha\ne 0$. So we use a constraint given by

$$x_1^2+\cdots+x_{p+1}^2=1 \tag{14}$$

(12) and (13) can be summarized as

$$F(x)\le 0 \tag{15}$$

where

$$F(x):=\begin{bmatrix} \hat f_1(x) & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \hat f_{q+1}(x)\end{bmatrix}$$

It is straightforward to show that $F$ can be always described of the form

$$F(x)=[x_1I,\ldots,x_{p+1}I]E\begin{bmatrix} x_1I \\ \vdots \\ x_{p+1}I\end{bmatrix}+E_{00}$$

for some $(q+1)\times(q+1)$ diagonal constant matrices $E_{ij}$, where

$$E := \begin{bmatrix} E_{11} & \dots & E_{1,p+1} \\ \vdots & \ddots & \vdots \\ E_{p+1,1} & \dots & E_{p+1,p+1} \end{bmatrix}$$

and $I$ is the $(q+1)\times(q+1)$ identity matrix. In the following arguments, without loss of generality, let $E_{ij} = E_{ji}, \forall i, \forall j$ be satisfied.

(15) is equivalent to

$$\lambda_{max}(F(x)) \le 0 \tag{16}$$

Combining (14) and (16), the SQI (4) is equivalent to a minimization problem given by

$$J = \min_{x^T x=1} \lambda_{max}(F(x)) \tag{17}$$

and the optimum value of $J$ is negative.

Furthermore, this is equivalent to a minimax problem given by

$$J = \min_{x^T x=1} \max_{y^T y=1} \phi(x,y) \tag{18}$$

where $\phi(x,y) := y^T F(x) y$ and $y = [y_1,\dots,y_{q+1}]^T \in \mathcal{R}^{q+1}$.

Considering from the viewpoint of two-person zero-sum game by two players $x$ and $y$, the minimax problem $J$ is the optimal strategy for the minimizer $x$. On the other hand, the optimal strategy for the maximizer $y$ is the maximin problem given by

$$J' = \max_{y^T y=1} \min_{x^T x=1} \phi(x,y)$$

In general $J \le J'$ holds [11], [12].

On the two-person zero-sum game, the following theorem by von Neumann is a classical result.

*Theorem 1 [11], [12]:* Assume that a saddle point $(x_0, y_0)$, i.e. a point that satisfies

$$h(x_0,y) \le h(x_0,y_0) \le h(x,y_0), \quad \forall x \in X, \forall y \in Y \tag{19}$$

exists for $h : X \times Y \to \mathcal{R}$, where $X$ and $Y$ are nonempty subsets of $\mathcal{R}^n$ and $\mathcal{R}^m$ and $n$ and $m$ are arbitrary integers. Then

$$\min_{x\in X}\max_{y\in Y} h(x,y) = h(x_0,y_0) = \max_{y\in Y}\min_{x\in X} h(x,y) \tag{20}$$

From Theorem 1, if a saddle point exists then that saddle point is an optimal solution to corresponding minimax problem.

Next we derive a condition that the saddle point for $\phi$ must satisfy.

The expression of $\phi$ by the power of $y$ is the definition of $\phi$ itself.

The expression of $\phi$ by the power of $x$ can be easily calculated as

$$\phi(x,y) = x^T G(y) x + \bar{g}(y) \tag{21}$$

where

$$\begin{aligned} G(y) &:= \begin{bmatrix} y^T E_{11} y & \dots & y^T E_{1,p+1} y \\ \vdots & \ddots & \vdots \\ y^T E_{p+1,1} y & \dots & y^T E_{p+1,p+1} y \end{bmatrix} && (22) \\ \bar{g}(y) &:= y^T E_{00} y && (23) \end{aligned}$$

Since $E_{ij} = E_{ji}, \forall i, j$ is satisfied, $G(y)$ is symmetric. Now we obtain the following theorem.

*Theorem 2:* $(x,y)$ is a saddle point of $\phi$ if and only if

$$\begin{aligned} \{\lambda I - F(x)\} y &= 0 && (24) \\ y^T y - 1 &= 0 && (25) \\ \lambda I - F(x) &\ge 0 && (26) \\ \{\rho I - G(y)\} x &= 0 && (27) \\ x^T x - 1 &= 0 && (28) \\ \rho I - G(y) &\le 0 && (29) \end{aligned}$$

*Proof:* If $x$ is fixed then the necessary and sufficient condition for $y$ to maximize $\phi$ is (24)-(26), i.e. $y$ is an eigenvector with respect to the largest eigenvalue of $F(x)$. If $y$ is fixed then the necessary and sufficient condition for $x$ to minimize $\phi$ is (27)-(29), i.e. $x$ is an eigenvector with respect to the smallest eigenvalue of $G(y)$. Q.E.D.

We have two comments on $\hat{f}_{q+1}$. (a) Under the constraints (13) and (14), $|z_i| < \sqrt{(1-\epsilon)/\epsilon}$ is always satisfied. Namely $\hat{f}_{q+1}$ has a property that makes the searching region of solution bounded. Therefore, even in case that the optimal solution is infinity (e.g. $f_i$'s are all first-order polynomial), the method proposed in this paper provides large but bounded solution. (b) In some cases, $x_{p+1} = 0$ cannot be the optimal solution to $J$. In such case, $\hat{f}_{p+1}$ can be removed from $F(x)$. See Section 6 for such example.

## IV. Continuous mapping and fixed point

In the followings, we propose a method to calculate the saddle point that satisfies (23)-(28). For this purpose, first we consider

$$\begin{aligned} \{\lambda I - (1-\theta)F_1 - \theta F(x)\}\eta + (1-\theta) f_0 &= 0 && (30) \\ \eta^T \eta - 1 &= 0 && (31) \\ \lambda I - (1-\theta)F_1 - \theta F(x) &\ge 0 && (32) \\ \{\rho I - (1-\theta)G_1 - \theta G(y)\} x + (1-\theta) g_0 &= 0 && (33) \\ x^T x - 1 &= 0 && (34) \\ \rho I - (1-\theta)G_1 - \theta G(y) &\le 0 && (35) \end{aligned}$$

where $\theta \in [0,1)$ is a constant and $f_0, F_1, g_0, G_1$ are arbitrary constant vectors and matrices of appropriate dimensions that satisfy $g_0 \ne 0$, $G_1 = G_1^T$, $F_1$ is a diagonal matrix and

$$f_0 = [f_{01},\dots,f_{0,q+1}]^T, \quad f_{0i} \ne 0, \quad \forall i. \tag{36}$$

(In the followings, we assume $f_{0,q+1} > 0$.)

Obviously, by setting $\theta = 1$ and replacing $\eta$ by $y$, (30)-(35) becomes the saddle point (24)-(29).

In the followings, we show that, at arbitrarily fixed $\theta \in [0,1)$, (30)-(32) can be recognized as a mapping from $x$ to $\eta$ (or $\eta'$ defined later) and (33)-(35) can be recognized as a mapping from $y$ (or $y'$ defined later) to $x$.

First, consider (30)-(32). Let $\theta \in (0,1)$ and $x$ be fixed. Then it is straightforward to show that (31) is equivalent to

$$\psi_1(\lambda) = 1 \tag{37}$$

where

$$\begin{aligned}\psi_1(\lambda) &:= (1-\theta)^2 f_0^T\{\lambda I-(1-\theta)F_1-\theta F(x)\}^{-2}f_0\\ &= (1-\theta)^2\sum_{i=1}^{q+1}\frac{f_{0i}^2}{\{\lambda I-(1-\theta)\hat{f}_{1i}-\theta\hat{f}_i(x)\}^2}\end{aligned}$$

and $\hat{f}_{1i}$ is $(i,i)$th-element of $F_1$. Since $f_{0i}(i=1,\ldots,q+1)$ satisfies (36), there exists unique $\lambda$ such that (37) and

$$\lambda > \max\{(1-\theta)\hat{f}_{11}+\theta\hat{f}_1(x),\ldots,(1-\theta)\hat{f}_{1,q+1}+\theta\hat{f}_{q+1}(x)\}$$

are satisfied. For this $\lambda$, $\{\lambda I-(1-\theta)F_1-\theta F(x)\}^{-1}$ exists. Using such $\lambda$, $\eta$ is uniquely determined as

$$\eta = -(1-\theta)\{\lambda I-(1-\theta)F_1-\theta F(x)\}^{-1}f_0 \tag{38}$$

Clearly as $x$ continuously moves, $\eta$ given by (38) also continuously moves.

Since $f_{0,q+1}>0$ as assumed earlier, from (38) obviously $\eta_{q+1}$ always satisfies

$$\eta_{q+1}<0 \tag{39}$$

So $\eta_{q+1}$ is always uniquely determined by

$$\eta_{q+1} = -\sqrt{1-\eta'^T\eta'} \tag{40}$$

where $\eta' := [\eta_1,\ldots,\eta_q]^T$. Of course $\eta'$ always satisfies $\eta'\in\mathcal{B}^q(:=\{x\in\mathcal{R}^n : x^Tx\le 1\})$.

Thus (30)-(32) can be recognized as a continuous mapping from $x$ to $\eta'$ (and $\eta_{q+1}$ is uniquely determined by (40)).

Next consider (33)-(35). Let $\theta\in(0,1)$ be fixed. Also, let $y'=[y_1,\ldots,y_q]\in\mathcal{B}^q$ be given and let $y_{p+1}$ be determined by $y_{q+1}=-\sqrt{1-y'^Ty'}$. Furthermore rewrite $G(y)$ as $G(y')$.

Next, define

$$\psi_2(\rho) := (1-\theta)^2 g_0^T\{\rho I-(1-\theta)G_1-\theta G(y')\}^{-2}g_0 \tag{41}$$

Now we consider the case that degeneration at the smallest eigenvalue of $(1-\theta)G_1+\theta G(y')$ (the situation that the smallest eigenvalue of $(1-\theta)G_1+\theta G(y')$ is not the pole of $\psi_2$) takes place in (41). Let $U$ be an orthogonal matrix that satisfy

$$\begin{cases} (1-\theta)G_1+\theta G(y') = U^T\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_{p+1}\end{bmatrix}U\\ (1-\theta)g_0 = U^T\begin{bmatrix}\beta_1\\ \vdots\\ \beta_{p+1}\end{bmatrix}\end{cases} \tag{42}$$

where $\lambda_1\le\lambda_2\le\cdots\le\lambda_{p+1}$.

Let the multiplicity of $\lambda_1$ be $r$, i.e. $\lambda_1=\cdots=\lambda_r<\lambda_{r+1}\le\cdots\le\lambda_{p+1}$ and define $\hat{\beta}:=[\beta_1,\ldots,\beta_r]^T$.

Then we have the following lemma.

*Lemma* : For almost all $g_0$, the Lebesgue measure of the set $\{y'\in\mathcal{B}^q : \hat{\beta}=0\}$ is zero.

*Proof* : Consider the space $\{(y',g_0) : y'\in\mathcal{B}^q, g_0\in R^{p+1}\}$. Let $y'$ be fixed. It follows from (42) that the Lebesgue measure of the set $\{g_0\in\mathcal{R}^{p+1} : \hat{\beta}=0\}$ is zero. Therefore the measure of $\{(y',g_0)\in\mathcal{B}^q\times\mathcal{R}^{p+1} : \hat{\beta}=0\}$ is also zero, since the measure of $\mathcal{B}^q$ is finite. Therefore the measure of $\{g_0\in\mathcal{R}^{p+1}$ : the measure of $y'$ that satisfy $\hat{\beta}=0$ is nonzero $\}$ must be zero. (If it is nonzero then the measure of $\{(y',g_0) : \hat{\beta}=0\}$ must be nonzero.) Q.E.D.

From this lemma, when $g_0$ is arbitrarily set, with probability one, $\{y'\in\mathcal{B}^q : \hat{\beta}=0\}$ is the set of measure zero. So, with probability one, $\hat{\beta}\ne 0$ is satisfied. Therefore, when $y'$ is given, we can determine the unique solution $x$ for (33)-(35) as

$$x = -(1-\theta)\{\rho I-(1-\theta)G_1-\theta G(y')\}^{-1}g_0 \tag{43}$$

where $\rho$ is the unique solution to

$$\begin{cases}\psi_2(\rho)=1\\ \rho I-(1-\theta)G_1-\theta G(y')<0\end{cases} \tag{44}$$

Clearly as $y'$ continuously moves, $x$ given by (43) also continuously moves.

Thus (33)-(35) can be recognized as a continuous mapping from $y'$ to $x$.

Now, we use Brouwer's fixed point theorem given as follows.

*Theorem 3 [14]:* For arbitrary $n$, any continuous mapping from $\mathcal{B}^n$ to $\mathcal{B}^n$ has at least one fixed point.

Using this theorem, we obtain the following theorem.

*Theorem 4:* There always exist $x$, $y$, $\lambda$ and $\rho$ that satisfy

$$\begin{aligned}\{\lambda I-(1-\theta)F_1-\theta F(x)\}y+(1-\theta)f_0 &= 0 \quad (45)\\ y^Ty-1 &= 0 \quad (46)\\ \lambda I-(1-\theta)F_1-\theta F(x) &\ge 0 \quad (47)\\ \{\rho I-(1-\theta)G_1-\theta G(y)\}x+(1-\theta)g_0 &= 0 \quad (48)\\ x^Tx-1 &= 0 \quad (49)\\ \rho I-(1-\theta)G_1-\theta G(y) &\le 0 \quad (50)\end{aligned}$$

for any fixed $\theta\in[0,1)$.

*Proof:* Let $\theta\in[0,1)$ be arbitrarily fixed. We consider (30)-(35) again.

As shown above, (30)-(32) can be recognized as a continuous mapping from $x\in\mathcal{R}^{p+1}$ to $\eta'\in\mathcal{B}^q$. We denote this mapping as $\eta'=\Phi_1(x)$.

Also, (33)-(35) can be recognized as a continuous mapping from $y'\in\mathcal{B}^q$ to $x\in\mathcal{R}^{p+1}$. We denote this mapping as $x=\Phi_2(y')$.

Combining them, we have

$$\eta' = \Phi(y') \tag{51}$$

where $\Phi := \Phi_1\circ\Phi_2$. Since $\Phi_1$ and $\Phi_2$ are continuous mapping, $\Phi$ is also continuous. So $\Phi$ is a continuous mapping from $\mathcal{B}^q$ to $\mathcal{B}^q$. Therefore, by using Brouwer's fixed point theorem, there exists $\eta'=y'\in\mathcal{B}^q$ that satisfy (51). Furthermore, from $y_{q+1}=-\sqrt{1-y'^Ty'}$ and $\eta_{q+1}=-\sqrt{1-\eta'^T\eta'}$, $y=\eta$ follows. By putting $y=\eta$ in (30)-(35), we obtain (45)-(50). Q.E.D.

Summarizing our argument so far, (30)-(35) can be recognized as a continuous mapping $\Phi$ from $y'\in\mathcal{B}^q$ to $\eta'\in\mathcal{B}^q$ for any fixed $\theta\in(0,1)$. A fixed point ( i.e. the point of $\eta=y$) of $\Phi$ is given by (45)-(50) for any fixed $\theta\in(0,1)$. By putting $\theta=1$ in the fixed point (45)-(50), we obtain the saddle point (24)-(29) of the minimax problem (18).

## V. USE OF HOMOTOPY

In this section, we aim to calculate the fixed point (45)-(50) of $\Phi$ for some fixed $\theta \in [0,1)$. This purpose can be achieved a homotopy method proposed in [8]. That homotopy is given by

$$(1-\hat{\theta})(y'-y'_0)+\hat{\theta}\{y'-\Phi(y')\}=0 \tag{52}$$

where $\hat{\theta} \in [0,1]$ is the homotopy parameter and $y'_0$ is a constant vector.

(52) is equivalent to

$$\begin{aligned}
(1-\hat{\theta})(y'-y'_0)+\hat{\theta}(y'-\eta') &= 0 && (53)\\
y^Ty-1 &= 0 && (54)\\
\{\lambda I-(1-\theta)F_1-\theta F(x)\}\eta+(1-\theta)f_0 &= 0 && (55)\\
\eta^T\eta-1 &= 0 && (56)\\
\lambda I-(1-\theta)F_1-\theta F(x) &\geq 0 && (57)\\
\{\rho I-(1-\theta)G_1-\theta G(y)\}x+(1-\theta)g_0 &= 0 && (58)\\
x^Tx-1 &= 0 && (59)\\
\rho I-(1-\theta)G_1-\theta G(y) &\leq 0 && (60)
\end{aligned}$$

It should be noted that $\theta$ and $\hat{\theta}$ are different. In (53)-(60), $\theta$ is a constant $\theta_0$ in $(0,1)$ and $\hat{\theta}$ is the homotopy parameter that moves from $\hat{\theta}=0$ to $\hat{\theta}=1$.

In this homotopy, the initial values $y_0, x_0, \eta_0, \rho_0, \lambda_0$ of variables $y, x, \eta, \rho, \lambda$ are set or calculated as follows. The initial value of $y'$ is $y'_0$, which is arbitrarily set as an interior point of $\mathcal{B}^q$. The initial value of $y_{q+1}$ is set as $y_{q+1}=-\sqrt{1-y'^Ty'}$. $\rho_0$ is determined by (58)-(60), and $x_0$ is determined by $\rho_0$ and (58). $\lambda_0$ is determined by (55)-(57), and $\eta_0$ is determined by $\lambda_0$ and (55).

As explained above, $y'_0$ is an interior point of $\mathcal{B}^q$, i.e. $y_0'^Ty'_0<1$. Also, (53) is $y'=(1-\hat{\theta})y'_0+\hat{\theta}\eta'$ and $\eta'^T\eta'\leq 1$ is satisfied. Therefore, $y'$ satisfies $y'^Ty'<1$ for any $\hat{\theta}\in[0,1)$. So, throughout $\hat{\theta}\in[0,1)$, $y_{q+1}\neq 0$. Therefore, since the initial value of $y_{q+1}$ is negative, throughout $\hat{\theta}\in[0,1)$, $y_{q+1}$ is negative, i.e. $y_{q+1}=-\sqrt{1-y'^Ty'}$.

It is guaranteed in [8] that tracking the homotopy path (53)-(60) reaches $\hat{\theta}=1$ with probability one. When this path-tracking reaches $\hat{\theta}=1$, we obtain $x, y, \lambda$ and $\rho$ that satisfy

$$\begin{aligned}
\{\lambda I-(1-\theta)F_1-\theta F(x)\}y+(1-\theta)f_0 &= 0 && (61)\\
y^Ty-1 &= 0 && (62)\\
\lambda I-(1-\theta)F_1-\theta F(x) &\geq 0 && (63)\\
\{\rho I-(1-\theta)G_1-\theta G(y)\}x+(1-\theta)g_0 &= 0 && (64)\\
x^Tx-1 &= 0 && (65)\\
\rho I-(1-\theta)G_1-\theta G(y) &\leq 0 && (66)
\end{aligned}$$

for some fixed $\theta=\theta_0\in(0,1)$. (61)-(66) is identical with (45)-(50).

Next we consider (61)-(66) as a homotopy in which $\theta$ is not constant but the homotopy parameter that moves from $\theta=\theta_0$ to $\theta=1$, and aim to obtain the saddle point (24)-(29).

After all, the whole calculation algorithm proposed in this paper is the following algorithm given by *Steps 1* to *3*. We call this algorithm Saddle-Point Homotopy Algorithm (SPHA). Also, we call (53)-(60) and (61)-(66) First-Stage Path-Tracking (FSPT) and Second-Stage Path-Tracking (SSPT), respectively.

*Step 1:* Execute the FSPT from $\hat{\theta}=0$ to $\hat{\theta}=1$, where $\theta$ is a fixed value $\theta_0\in(0,1)$.

*Step 2:* After the FSPT reaches $\hat{\theta}=1$, excecute the SSPT from $\theta=\theta_0$ to $\theta=1$.

*Step 3:* If the SSPT in *Step 2* reaches $\theta=1$, the saddle point (24)-(29) is obtained. If the SSPT in *Step 2* does not reach $\theta=1$, change the value of $\theta_0$ closer to 1 and repeat *Step 1* and *Step 2* again.

When we execute FSPT and SSPT, how to deal with inequalities (57) and (60) (or (63) and (66)) can be chosen from the following two types. (We explain about FSPT. Explanation for SSPT is similar.)

*Type A:* We track the path constructed by (53)-(56), (58) and (59) only. Along this tracking, we continue to check whether (57) and (60) are satisfied or not. When we detect that (57) or (60) is not satisfied, we stop the path-tracking temporarily, and return to $\hat{\theta}$ for which (57) and (60) are satisfied, and return to the path-tracking again with smaller increment of $\hat{\theta}$.

*Type B:* We replace inequalities (57) and (60) by equalities

$$\lambda I-(1-\theta)F_1+\theta F(x)-V^TV=0 \tag{67}$$

and

$$\rho I-(1-\theta)G_1-\theta G(y)+W^TW=0 \tag{68}$$

where $V$ and $W$ are new unknowns that have the form of upper-right matrices They use so-called Choresky factorizations (e.g. [16]).

Now we obtain the following theorem.

*Theorem 5:* By the SPHA, with probability one the saddle point (24)-(29) is obtained.

*Proof:* First, it is guaranteed in [8] that the FSPT (53)-(60) reaches $\hat{\theta}=1$ with probability one.

Next consider the SSPT. Path (61)-(66) is a plot of fixed point as $\theta$ varies. This path may not be continuous between $\theta=0$ and $\theta=1$, and may consist of more than two curves. However, even if the path consists of more than two curves, it follows from Theorem 4 that at least one such curve reaches $\theta=1$. Therefore, by making the value of $\theta_0$ sufficiently close to 1, the end of SSPT must reach the path that is connected to the plane $\theta=1$. By executing the SSPT on that curve, we can always reach $\theta=1$. Q.E.D.

After all, our result on nonconvex optimization problem is as follows:

*Theorem 6:* By the SPHA corresponding to the nonconvex optimization problem (1)-(3), with probability one the global optimum of the nonconvex optimization problem (1)(3) is obtained after $\gamma$ iteration.

The calculation method proposed in this note can be used for the problem of finding a numerical solution to the set of algebraic equations and some problems in discrete mathematics. See Appendices II and III for these topics.

## VI. NUMERICAL EXAMPLES

In this section, we consider a numerical example of nonconvex optimization problem given by

$$J=(z-a_1)^2+a_2\rightarrow\min \tag{69}$$

in a nonconvex region

$$-z^2 + 1 \leq 0 \tag{70}$$

Using $z = x_1/x_2$ and $\gamma$, (69) and (70) are equivalent to

$$\begin{aligned} x_1^2 - a_1 x_1 x_2 + (a_1^2 + a_2 - \gamma)x_2^2 &\leq 0 \quad (71) \\ -x_1^2 + x_2^2 &\leq 0 \quad (72) \end{aligned}$$

Obviously $x_2 = 0 (z = \infty$ or $-\infty)$ does not correspond to the global optimum of this problem. So we can remove (13) from $F(x)$.

Therefore in this example

$$F(x_1, x_2) = \begin{bmatrix} x_1^2 - a_1 x_1 x_2 + (a_1^2 + a_2 - \gamma)x_2^2 & 0 \\ 0 & -x_1^2 + x_2^2 \end{bmatrix} \tag{73}$$

and

$$\begin{aligned} E_{11} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, E_{12} = E_{21} = \begin{bmatrix} -a_1 & 0 \\ 0 & 0 \end{bmatrix}, \\ E_{22} &= \begin{bmatrix} a_1^2 + a_2 - \gamma & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{74}$$

We use the Type A. For example we put

$$F_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, f_0 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, G_1 = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}, g_0 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \tag{75}$$

and $a_2 = 3$.

We investigate two cases $a_1 = -2$ and $a_1 = 0.5$.

*Case 1:* $a_1 = -2$. In this case, the problem has the global minimum at $z = -2$ and a local minimum at $z = 1$. After $\gamma$-iteration, the value of $\gamma$ approaches $a_2 (= 3)$.

First we execute the FSPT from $\hat{\theta} = 0$ to $\hat{\theta} = 1$ where $\theta = 0.8$ is fixed and $\gamma = 3$. As a result of this tracking, we can confirm the path is continuous between the initial point

$$\begin{aligned} \xi_1 = {} & [0.61617, -0.78762, 1.53069, -0.74374, 0.66847, \\ & -1.87280, 0.60000, -0.8000, 0.000]^T \end{aligned} \tag{76}$$

and the final point

$$\begin{aligned} \xi_1 = {} & [0.61568, -0.78800, 1.52661, -0.74524, 0.66679, \\ & -1.86777, 0.61575, -0.78794, 1.000]^T. \end{aligned} \tag{77}$$

where $\xi_1 = [\eta_1, \eta_2, \lambda, x_1, x_2, \rho, y_1, y_2, \hat{\theta}]^T$.

Next we execute the SSPT from $\theta = 0.8$ to $\theta = 1$. As a result of this tracking, we can confirm the path is continuous between the initial point

$$\begin{aligned} \xi_2 = {} & [0.61575, -0.78794, 1.52661, -0.74524, \\ & 0.66679, -1.86777, 0.800]^T \end{aligned} \tag{78}$$

and the final point

$$\begin{aligned} \xi_2 = {} & [1.000, 7.0 \times 10^{-7}, -2.6 \times 10^{-7}, -0.89443 \\ & 0.44721, 5.6 \times 10^{-7}, 1.000]^T. \end{aligned} \tag{79}$$

where $\xi_2 = [y_1, y_2, \lambda, x_1, x_2, \rho, \theta]^T$.

It follows from the final point (79) that

$$z = \frac{x_1}{x_2} = \frac{-0.89443}{0.44721} = -2.0000 \tag{80}$$

This corresponds to the global minimum.

*Case 2:* $a_1 = 0.5$. In this case, the problem has the global minimum at $z = 1$ and a local minimum at $z = -1$. After $\gamma$-iteration, the value of $\gamma$ approaches $(1 - a_1)^2 + a_2 (= 3.25)$.

First we execute the FSPT from $\hat{\theta} = 0$ to $\hat{\theta} = 1$ where $\theta = 0.8$ is fixed and $\gamma = 3.25$. As a result of this tracking, we can confirm the path is continuous between the initial point

$$\begin{aligned} \xi_1 = {} & [0.31737, -0.94830, 1.74050, -0.56076, 0.82798, \\ & -2.19328, 0.60000, -0.8000, 0.000]^T \end{aligned} \tag{81}$$

and the final point

$$\begin{aligned} \xi_1 = {} & [0.44492, -0.89557, 1.49712, 0.70541, 0.70880, \\ & -2.24718, 0.44415, -0.89515, 1.000]^T. \end{aligned} \tag{82}$$

Next we execute the SSPT from $\theta = 0.8$ to $\theta = 1$. As a result of this tracking, we can confirm the path is continuous between the initial point

$$\begin{aligned} \xi_2 = {} & [0.44415, -0.89515, 1.49712, 0.70541, \\ & 0.70880, -2.24718, 0.800]^T \end{aligned} \tag{83}$$

and the final point

$$\begin{aligned} \xi_2 = {} & [0.81650, -0.57735, 7.8 \times 10^{-8}, 0.70711 \\ & 0.70711, 2.6 \times 10^{-8}, 1.000]^T. \end{aligned} \tag{84}$$

It follows from the final point (84) that

$$z = \frac{x_1}{x_2} = \frac{0.70711}{0.70711} = 1.0000 \tag{85}$$

This corresponds to the global minimum.

## VII. Conclusions

In this paper, new numerical method for finding the global optimum of polynomial-type nonconvex optimization problem has been proposed Our key tool is the combination of probability-one homotopy method and minimax theorem. By using our method, the global, not local, optimum of polynomial-type nonconvex optimization problem can be calculated after $\gamma$ iteration with probability one.

In our problem formulation, $f_i^{NC}$ are restricted to be polynomials. Removing this restriction is our future subject of research.

The result on BMI is also our future subject.

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## APPENDIX A
## TRANSFORMATION TO SQI

For example, we consider an inequality given by

$$z_1^2 - 3z_1z_2^2z_3 \leq 0, \tag{86}$$

By introducing new unknowns $z_4$ and $z_5$ as

$$z_4 := z_2^2 \tag{87}$$
$$z_5 := z_4z_3 (= z_2^2z_3) \tag{88}$$

(86) becomes

$$z_1^2 - 3z_1z_5 \leq 0. \tag{89}$$

and are equivalent to

$$z_4 - z_2^2 = 0 \tag{90}$$
$$z_5 - z_4z_3 = 0 \tag{91}$$

After all, (86) is equivalent to (89)-(91) The order of left-hand side of (89)-(91) are less than or equal to 2.

Similarly as this example, in any case, by replacing second-order term by new unknown, (1)-(3) is always equivalently transformable to (7).

## APPENDIX B
## SET OF ALGEBRAIC EQUATIONS

The problem of finding the numerical solution to a set of algebraic equations (SAE) $f_1(z_1, \ldots, z_n) = \cdots = f_m(z_1, \ldots, z_n) = 0$ is equivalent to the problem of finding the global minimum of minimization problem $f_1^2 + \cdots + f_m^2 \to \min$. The latter is a nonconvex minimization problem in general. So the method proposed in this note can be applied to it. (Also, it is easy to equivalently transform a given SAE to a SQI by using the method shown in Section 2.)

## APPENDIX C
## PROBLEM IN DISCRETE MATHEMATICS

Here we consider prime factorization. Factorization of a given odd number $\alpha$ is equivalent to

$$z_1z_2 = \alpha \tag{92}$$
$$z_i = z_{i0} + 2z_{i1} + 2^2z_{i2} + \cdots + 2^nz_{in} \tag{93}$$
$$z_{ij} \in \{0, 1\} \tag{94}$$
$$i = 1, 2; j = 0, \ldots, n$$

where $n$ is an integer that satisfy $2^n \leq \alpha/3 < 2^{n+1}$.

Furthermore (94) is equivalent to

$$z_{ij}(z_{ij} - 1) = 0 \tag{95}$$

Since $2^{n+1} \leq (2/3)\alpha < \alpha$ is satisfied and $z_i$ is given by (93), $z_i = \alpha$ cannnot happen. So the possibility of $(z_1, z_2) = (1, \alpha)$ or $(\alpha, 1)$ is automatically removed.

Clearly (92),(93) and (95) are the set of algebraic equation. So we can use the method proposed in this note for finding the numerical solution to it.