Tsunami Solitons Emerging from Superconducting Gap

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We propose a classical integrable system exhibiting the tsunami-like solitons with rocky-desert-like disordered stationary background. One of the Lax operators describing this system is interpretable as a Bogoliubov-de Gennes Hamiltonian in parity-mixed superconductor. The family of integrable equations is generated from this seed operator by Krichever's method, whose pure s-wave limit includes the coupled Schrödinger-Boussinesq hierarchy applied to plasma physics. A linearly unstable finite background with superconducting gap supports the tsunami-soliton solution, where the propagation of the step structure turns back at a certain moment, accompanied with the oscillation on the opposite side. In addition, the equation allows inhomogeneous stationary solutions with arbitrary number of bumps at arbitrary positions, which we coin *the KdV rocks*. In the Zakharov-Shabat scheme, the tsunami solitons are created from the Bogoliubov quasiparticles in energy gap and the KdV rocks from normal electrons/holes. The unexpected large space of stationary solutions comes from the non-coprime Lax pair and the multi-valued Baker-Akhiezer functions on the Riemann surface, formulated in terms of higher-rank holomorphic bundles by Krichever and Novikov. Furthermore, the concept of *isodispersive phases* is introduced to characterize quasiperiodic multi-tsunami backgrounds and consider its classification.

The study of rogue waves has brought a new interdisciplinary trend between classical integrable systems and nonlinear physics.[1–3] As a prime example, the attractive nonlinear Schrödinger (NLS) equation has three kinds of fundamental solitons; spatially localized and temporally oscillating Kuznetsov-Kawata-Inoue-Ma soliton,[4–6] temporally localized and spatially periodic Akhmediev breather (AB),[7,8] and spatiotemporally localized Peregrine soliton,[9] the most famous rogue wave. Many integrable systems with modulational instability also exhibit similar solutions.[10–12]

The statistical properties of integrable turbulence has also been investigated from the view of rogue wave formation.[13] Other notable wave phenomena studied in integrable systems are the dispersive shock waves based on Whitham's method.[14] Under these circumstances, the question arises as to whether it is possible to describe a wider range of hydrodynamic instability phenomena within the frame work of exact analysis of classical integrable systems. The aim of this paper is to propose the classical integrable equation possessing tsunami-like soliton solutions propagating on bumpy and disordered backgrounds, emerging, surprisingly, as a by-product of the study of superconductivity.

Our system is described by the Lax pair

$$i\hat{L}_t = [\hat{L}, \hat{M}], \tag{1}$$

$$\hat{L} = -\partial_x^2\sigma_3 + \{\partial_x, \xi\sigma_+ - \eta\sigma_-\} - u\sigma_3 + v\mathbf{1}_2 + q\sigma_+ + r\sigma_-, \tag{2}$$

$$\hat{M} = (-\partial_x^2 - u + 2\xi\eta)\mathbf{1}_2 + v\sigma_3 + 2(\xi_x\sigma_+ + \eta_x\sigma_-), \tag{3}$$

where $\mathbf{1}_2$ and $\sigma_{1,2,3}$ are the 2×2 identity and Pauli matrices, $\sigma_\pm = \frac{\sigma_1 \pm i\sigma_2}{2}$, and $\{X, Y\}$ is the anti-commutator. Here, we restrict ourselves to the case $u = u^*$, $v = v^*$, $r = q^*$, and $\eta = \xi^*$, where $\hat{L}$ and $\hat{M}$ become self-adjoint. The resultant equations are given by

$$iu_t = 2(q^*\xi_x - q\xi_x^*), \quad iv_t = 2(\xi\xi_x^* - \xi^*\xi_x)_x, \tag{4}$$

$$i\xi_t = -2v\xi + q_x, \tag{5}$$

$$iq_t = -2vq - 2\xi u_x - 4\xi_x u + 4\xi(|\xi|^2)_x - \xi_{xxx}. \tag{6}$$

The physical/mathematical backgrounds which led us to Eqs. (1)-(6) are to be confessed below.

First, $\hat{L}$ [Eq. (2)] can be viewed as the Bogoliubov-de Gennes (BdG) Hamiltonian[15,16] in parity-mixed superconductors (SCs) and physical interpretations of the coefficient functions by the Hartree-Fock (HF) mean fields and the Cooper pairs (gap functions) are summarized in Table I. The mean-field theory with omitted HF fields is formulated in Ref. 17. The parity-mixed SCs appear in noncentrosymmetric materials[18] and the surface of topological SCs.[19] Figure 1 shows a schematic picture of the dispersion relation with/without s-wave gap $q = q_0$.

When chemical potential μ is large, Andreev's dispersion linearization[20] around the Fermi points (see Fig. 1 of Ref. 21) works well and physical systems equivalent to this one appear in diverse fields, including the Peierls problem in conducting polymers[22–30] and the Gross-Neveu models,[31–33] and the reflectionless/finite-gap potentials in the NLS hierarchy have been applied to determine the exact self-consistent soliton dynamics and the phase diagram of the Larkin-Ovchinnikov (LO) and Fulde-Ferrell (FF) soliton lattices.[34–43]

On the other hand, the treatment *without* dispersion linearization become important in (i) the BCS-BEC crossover[44,45] where the BEC side corresponds to small μ, (ii) eliminating cutoff-dependence, and (iii) comparsion with quantum many-body counterpart described by the Gaudin-Yang model.[46–49] In particular, (iii) will play a key role in construction of the fermionic many-body quantum solitons, whose bosonic analog has been studied.[50–55] The quadratic dispersion also contributes to spectral functions and lifetime of quasiparticles in Tomonaga-Luttinger liquids.[56,57]

Next, we explain how to find $\hat{M}$ in Eq. (3). Following Krichever,[58] we can find a family of differential operators $\hat{M}_{n,\pm}$, $n = 0, 1, 2, \ldots$ commuting with $\hat{L}$, and obtain the sequence of ordinary differential equations determined by $[\hat{L}, \hat{M}] = 0$, called the Novikov equation. $\hat{M}_{n,\pm}$ is defined by

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Table I. The physical interpretation of the potentials in $\hat{L}$, describing the parity-mixed SC. Here, $\hat{\psi}_s$, $s=\uparrow,\downarrow$ represent fermionic field operators with up and down spin, g and g' are s- and p-wave coupling constants, and the bracket means the expectation value for a certain quantum many-body state.

	Interpretation in SCs	
u	$\mu+\frac{1}{2}g\langle\hat{\psi}_\uparrow^\dagger\hat{\psi}_\uparrow+\hat{\psi}_\downarrow^\dagger\hat{\psi}_\downarrow\rangle$	chemical potential + HF mean field
v	$-h+\frac{1}{2}g\langle\hat{\psi}_\uparrow^\dagger\hat{\psi}_\uparrow-\hat{\psi}_\downarrow^\dagger\hat{\psi}_\downarrow\rangle$	magnetic field + HF mean field
q	$g\langle\hat{\psi}_\downarrow\hat{\psi}_\uparrow\rangle$	s-wave Cooper pair
ξ	$g'\langle\hat{\psi}_\uparrow\partial_x\hat{\psi}_\downarrow+\hat{\psi}_\downarrow\partial_x\hat{\psi}_\uparrow\rangle$	p-wave Cooper pair

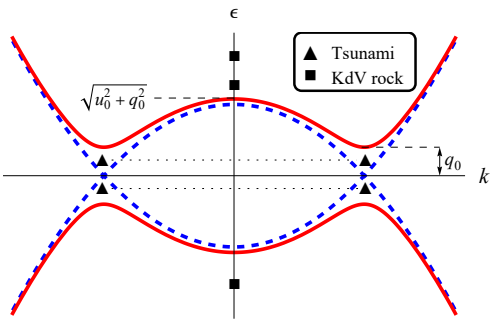

Fig. 1. The dispersions relation $\epsilon(k)$ for plane-wave eigenfunctions $\hat{L}\phi=\epsilon\phi$, $\phi\propto e^{ikx}$, with uniform background $(u,v,q,\xi)=(u_0,0,q_0,0)$. The red solid and blue dashed lines represent the relation with ($q_0\neq 0$) and without ($q_0=0$) the s-wave superconducting gap. The triangle and square markers show exponentially-divergent seed solutions used to construct soliton solutions.

its highest-order term $\hat{M}_{n,+}=(-\mathrm{i}\partial_x)^n\mathbf{1}_2+\cdots$ and $\hat{M}_{n,-}=(-\mathrm{i}\partial_x)^n\sigma_3+\cdots$. If commutators are proportional to the time derivative $\hat{L}_t$, we obtain the hierarchy of classical integrable systems. Here we use $\hat{M}=\hat{M}_{2,+}$ in Eq. (3), because it provides the lowest-order equation supporting the stationary background with superconducting gap shown in Fig. 1. The *Mathematica* code for generating Novikov equations is available,[59,60] including expressions up to $\hat{M}_{5,\pm}$.

When $v=\xi=0$, the above hierarchy reduces to the one including the coupled Schrödinger-Boussinesq equation,[12,61–66] which has been applied to plasma physics. Therefore, while the time evolution governed by $\hat{M}$ is different from those in condensed-matter systems mentioned above whose time evolution is based on the self-consistent determination of potentials and eigenfunctions[17,41–44,67] or effective field theories,[68] the present equations (4)-(6) are expected to be derived by applying the reductive perturbation method[61,62,69–71] to multicomponent plasmas or fluids.

Now, let us move on to the construction of the concrete multi-soliton solutions by the Zakharov–Shabat (ZS) scheme.[72] We extract the minimal formulas for the higher-order ZS scheme:[73] Let w_i, $i=1,\ldots,n$ be an eigenfunction of the Lax pair $(\hat{L}-\epsilon_i)w_i=(\hat{M}+\mathrm{i}\partial_t)w_i=0$ with asymptotic behavior $\|w_i\|\to 0$ (resp. ∞) at $x\to-\infty$ (resp. $+\infty$), which are called the seed solution. Writing their array as $W=(w_1,w_2,\ldots,w_n)$, we introduce an $n\times n$ Gram matrix $G(x)=\int_{-\infty}^{x}dxW(x)^\dagger W(x)$, and define $K(x,y)=-W(x)[I_n+G(x)]^{-1}W(y)^\dagger$. Then, the new solution with n solitons added from the known one is given by

$$\xi^{\mathrm{new}}=\xi+K_{12}, \tag{7}$$

$$q^{\mathrm{new}}=q-2K_{12}K_{22}+(K_x-K_y)_{12}+2\xi(K_{11}-K_{22}), \tag{8}$$

$$u^{\mathrm{new}}=u-2|\xi|^2+2|\xi^{\mathrm{new}}|^2+[\ln\det(I_n+G)]_{xx}, \tag{9}$$

$$v^{\mathrm{new}}=v+\mathrm{tr}[(K_x+K_y)\sigma_3], \tag{10}$$

where $K=K(x,x)$, $K_x=[\partial_xK(x,y)]_{y=x}$, $K_y=[\partial_yK(x,y)]_{y=x}$ and subscripts 1,2 represent the matrix components. The formulas (7)-(10) might be expressed elegantly using quasideterminant.[74]

Let us apply the above general formula to the uniform state with s-wave gap $(u,v,q,\xi)=(u_0,0,q_0,0)$, $u_0,q_0>0$. The eigenfunction with real eigenvalue ϵ is given by

$$w(x,t,\epsilon,k,\varphi)=\sqrt{2|\,\mathrm{Re}\,k|}\left(w_0(\epsilon,k)e^{kx-\omega(k)t+i\varphi}+\mathrm{c.c.}\right), \tag{11}$$

where $w_0(\epsilon,k)=\frac{1}{\sqrt{2}}\frac{1}{[(\epsilon+q_0)^2+|\omega(k)|^2]^{1/2}}\begin{pmatrix}\epsilon+q_0+i\omega(k)\\ \epsilon+q_0-i\omega(k)\end{pmatrix}$, $\omega(k)=i(k^2+u_0)$, and $k=k(\epsilon):=[(\epsilon^2-q_0^2)^{1/2}-u_0]^{1/2}$. Then, the seed solution is generally given by $w_i=w(x-x_i,t-t_i,\epsilon_i,k_i,\varphi_i)$, $k_i=k(\epsilon_i)$, possessing four real parameters x_i,t_i,φ_i and ϵ_i. The soliton velocity becomes $V_i=\frac{\mathrm{Re}\,\omega(k_i)}{\mathrm{Re}\,k_i}$. Below, we find the two types of one-soliton solution by the choice of ϵ_1 (Fig. 1).

The first type is found from the superconducting gap $|\epsilon|<q_0$ (Fig. 1), which we call the *tsunami* soliton, whose behavior is shown in Fig. 2. In this solution, we observe the propagation of the step structure, which suddenly turns back at $(x,t)=(x_1,t_1)$. At the moment of turning back, the opposite side of soliton experiences an oscillation.

Taking various limits of (x_1,t_1), we can obtain different solutions. If both x_1,t_1 are set infinity, we get a solution without turning back. If we fix t_1 and take $x_1\to-\infty$, we obtain a solution with sudden oscillation occurring at $t=t_1$, similar to the AB, but now emerges from the self-adjoint Lax pair. We note the unpredictability of the moment of turning back from the observation data — it depends on a very subtle difference of the initial condition and difficult to detect. Unlike rogue waves,[13] the background with finite u_0,q_0 is linearly unstable against a *short*-wavelength perturbation; the onset of instability appears at $k=2k_F=2\sqrt{u_0}$ with k_F a Fermi wavenumber, which explains the oscillation period occurring at $t=0$ in Fig. 2. The $2k_F$-oscillation around a local defect is called the Friedel oscillation in condensed-matter context.[75,76]

Figure 2 also reminds us of the soliton resonance phenomena in (2+1)-dimensional integrable systems,[71,77,78] where Y-shaped and more divaricate structures of line solitons are formed, using large degree of freedom including functional parameters.[79] Both are constructed by linear combination of multiple seed solutions, but in the present case, the maximum number of seeds is two [Eq. (11)] due to limitation of (1+1)-dimensional system. The oscillation profile similar to Fig. 2 can be found in two-layer fluid.[71]

In Fig. 2, we plot $u-2|\xi|^2$ instead of u itself, because it is a conserved density and convenient to detect the front of the tsunami soliton. The conservation laws are derived as follows.[80] Let Φ be a 2×2 matrix satisfying $(\hat{L}-\lambda)\Phi=(\hat{M}+\mathrm{i}\partial_t)\Phi=0$. Defining $\Psi=\begin{pmatrix}\Phi\\ \partial_x\Phi\end{pmatrix}$, we have a zero-curvature expression $\partial_x\Psi=U\Psi$, $\partial_t\Psi=V\Psi$ using some λ-dependent 4×4 matrices U and V. Putting $\lambda=k^2$ and expanding the Riccati equation for $\Gamma=\Phi_x\Phi^{-1}$ by powers $(\frac{1\mp i}{k})^n$, we obtain

$$\frac{\partial}{\partial t}(\rho_n^{\mathrm{R}}\pm i\rho_n^{\mathrm{I}})+\frac{\partial}{\partial x}(J_n^{\mathrm{R}}\pm iJ_n^{\mathrm{I}})=0,\quad n=1,2,\ldots, \tag{12}$$

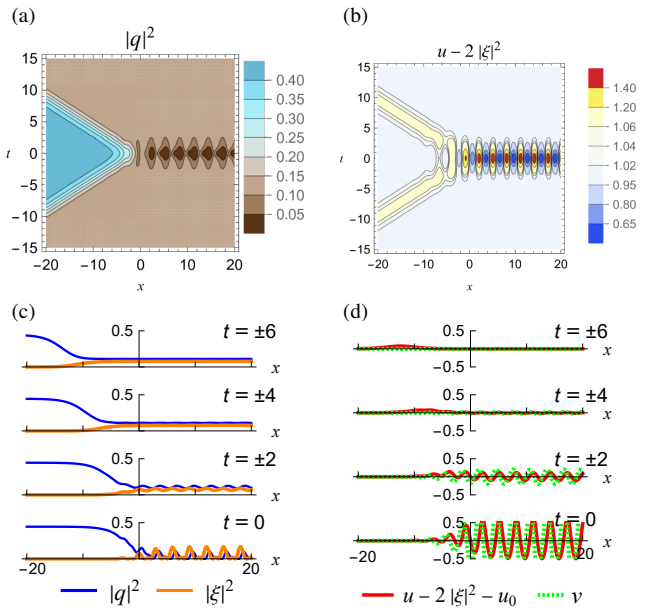

Fig. 2. One-tsunami solution. The background parameters are $(u_0, q_0) = (1, \frac{2}{3})$ and soliton parameters are $(\epsilon_1; x_1, t_1, \varphi_1) = (\frac{1}{3}; 0, 0, 0)$. (a) and (b) are contour plots for $|q|^2$ and $u-2|\xi|^2$. (c) and (d) show snapshots at special t's for $|q|^2, |\xi|^2, u-2|\xi|^2-u_0$, and v. The gif animation for (c) and (d) is available.[59,81]

whose first few examples are $\rho_1^{\rm R} = -v$, $\rho_1^{\rm I} = u - 2|\xi|^2$, $\rho_2^{\rm R} = 2(|\xi|^2)_x$, $\rho_2^{\rm I} = -v_x$, ..., and $J_1^R = 2i(\xi^*\xi_x - \xi\xi_x^*)$, $J_1^{\rm I} = 2i(q^*\xi - q\xi^*)$, $J_2^{\rm R} = 2i(\xi^* q_x - \xi q_x^*)$, $J_2^{\rm I} = 2i(\xi^*\xi_{xx} - \xi\xi_{xx}^*)$, ... etc. For stationary solution, $J_n^{\rm R,I}$'s are constant and appear as coefficients of the algebraic curve.[58]

The second type of solitons arises from the bound states of normal electrons/holes (Fig. 1), which has zero velocity. We call them *the Korteweg-de Vries (KdV) rocks*, because $\hat{L}$ with $q = \xi = 0$ reduces to the double Schrödinger operator, and hence the time evolution base on the *third order* $\hat{M}_{3,\pm}$ *instead of* $\hat{M}_{2,+}$ becomes the famous KdV equation. They are immobile under $\hat{M}_{2,+}$, but *could be* a moving KdV soliton if time evolution *were* defined by $\hat{M}_{3,\pm}$. The hybrid multi-soliton solution with coexisting tsunami solitons and KdV rocks is shown in Fig. 3, like a flood in the desert.

The lack of time-dependence for the KdV rocks does not mean that this solution is boring — the fact that we can add arbitrary number of the stationary KdV rocks at any position is curious, because it allows the present system (4)-(6) to have *anomalously* large stationary-solution space. This should be compared with the typical known classical integrable systems, where general stationary solutions are given by elliptic functions and the whole solution space has at most a finite number of adjustable constants. As we will see below, this anomaly can happen because the orders of $\hat{L}$ and $\hat{M}$, which are now both two [Eqs. (2) and (3)], are *not* coprime. Recall that, in the KdV hierarchy, the only *odd-order* differential operators generate the higher-order KdV equations,[82,83] so there is no counterpart for *even-order* $\hat{M}_{n,\pm}$'s.

Consider the stationary problem for Eq. (1), i.e., $[\hat{L}, \hat{M}] = 0$. By the Burchnall-Chaundy lemma,[58,83–87] the commuting differential operators satisfy a polynomial relation $P(\hat{L}, \hat{M}) = 0$ which defines an algebraic curve (or Riemann surface). Writing the simultaneous eigenvalue problem

$$\hat{L}\phi = \lambda\phi, \ \hat{M}\phi = \omega\phi, \tag{13}$$

the algebraic curve satisfied by (2) and (3) is

$$P(\lambda, \omega) = (\lambda^2 - \omega^2 - a)^2 + b\lambda + c\omega + d = 0, \tag{14}$$

where a, b, c, d are expressed by rational functions of the constants $J_n^{\rm R,I}$'s in Eq. (12). If the values of a, b, c, d are generic, Eq. (14) represents the genus-one elliptic curve. The stationary solutions are then divided into two classes, which we call *regular* and *irregular* below.

The *regular* solutions are described as follows. Assume that $J_1^{\rm R} \neq 0$ and write $\alpha = J_1^{\rm I}/J_1^{\rm R}$ and $\beta_3 = J_3^{\rm R}/J_1^{\rm R}$. The stationary solution is then given by $q = -\alpha\xi_x$, $u \pm v = \frac{1\pm\alpha}{1\mp\alpha}(\beta_3 - 2|\xi|^2)$, and $(1-\alpha^2)\xi_{xx} + 4(\beta_3 - 2|\xi|^2)\xi = 0$. The last one is the stationary NLS equation whose solution is given by theta functions.[88,89] The algebraic curve (14) for regular solutions has genus $g = 1$ except for elementary limit. These results are familiar and typical.

The *irregular* solutions, on the other hand, emerge when $J_1^{\rm R} = J_1^{\rm I} = 0$. We have general solution $\xi^* = c_1\xi$, $q^* = c_1 q$, $v = \frac{q_x}{2\xi}$, and $u = \frac{c_2}{\xi^2} - \frac{q^2}{4\xi^2} + c_1\xi^2 + \frac{\xi_x^2}{4\xi^2} - \frac{\xi_{xx}}{2\xi}$, where c_1, c_2 are integration constants. Thus, the irregular solutions have two *arbitrary* real-valued (up to overall phase) functions ξ and q, which include the multi-KdV-rock states as a particular solution. This result also suggests that the irregular solution need not generally be a reflectionless potential. With this solution, the algebraic curve (14) has the squared form

$$P(\lambda, \omega) = (\lambda^2 - \omega^2 - 4c_1c_2)^2 = 0, \tag{15}$$

suggesting the double-valuedness of the Baker-Akhiezer (BA) function discussed below.

If orders of $\hat{L}$ and $\hat{M}$ are coprime, the BA function, i.e., the simultaneous eigenfunction ϕ in Eq. (13) defined as a function on the Riemann surface, becomes a *single-valued* meromorphic (excepting the essential singular point) function, whose uniqueness under a certain condition guarantees the algebro-geometric approach.[58,83,90,91] If not, however, they *can* be (but not always) a *multi-valued* function, and regarded as a section of the higher-rank holomorphic vector bundle.[86,92,93] In such a case, the solution of the differential equation may have arbitrariness of functional parameters.[93]

To illustrate the above theories, let us observe that the spatially uniform state belonging to *regular* solutions does not allow a time-independent KdV rock. Take the FF state $(u \pm v, q, \xi) = (p^2(1\pm\alpha)^2, -2i\alpha p\xi_0 e^{2ipx}, \xi_0 e^{2ipx})$, where p, ξ_0, α are real. We only discuss $\alpha = 0$ for brevity and assume $p \neq 0$. The eigenfunction is obtained by substituting $\phi \propto e^{i(k+p\sigma_3)x}$ to Eq. (13), yielding the dispersion relations $(\lambda - 2kp)^2 = k^4 + 4\xi_0^2 k^2$, $(k^2 + 2\xi_0^2 - \omega)^2 = 4p^2(k^2 + 4\xi_0^2)$. Eliminating k from these two relations, we revisit a relation of the form (14), whose genus is now zero because of elementary limit, parametrized by the uniformization parameter s: $(\lambda, \omega) = (X+Y, X-Y)$, $X = 4p^2 s(s-1)$, $Y = \frac{\xi_0^2}{s} - \frac{\xi_0^4}{4p^2 s^2}$. Moreover, *the wavenumber k is also written as a rational function of the same parameter:* $k = -2ps + \frac{\xi_0^2}{2ps}$, implying that the BA function ϕ is *single-valued* on the Riemann surface. Indeed, $\omega = -\lambda - 4i|p|\xi_0^2\lambda^{-1/2} + O(\lambda^{-1})$ for large λ has both nonzero real and imaginary parts, so the KdV soliton created by the ZS scheme has finite velocity, no longer a "rock". Hence, no

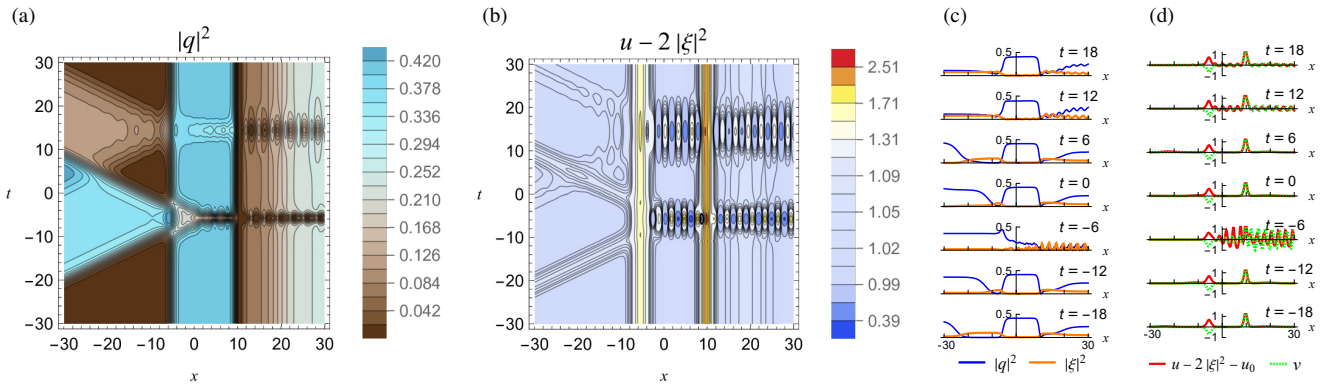

Fig. 3. Five-soliton solution with two tsunami solitons and three KdV rocks.[81] The plotted objects are the same as Fig. 2. The parameters are $(u_0, q_0) = (1, \frac{2}{3})$ for background, $(\epsilon_1; x_1, t_1, \varphi_1) = (\frac{1}{3}; 0, -5, 0)$, $(\epsilon_2; x_2, t_2, \varphi_2) = (\frac{3}{5}; -3, 13, 0)$ for tsunami solitons, and $(\epsilon_3; x_3, t_3, \varphi_3) = (-\frac{9}{5}; -5, 0, \frac{\pi}{6})$, $(\epsilon_4; x_4, t_4, \varphi_4) = (\frac{5}{2}; 10, 0, \frac{\pi}{6})$, $(\epsilon_5; x_5, t_5, \varphi_5) = (\frac{5}{4}; 19, 0, \frac{\pi}{6})$ for KdV rocks. The gif animation for (c) and (d) is available.[59,81]

large arbitrariness in stationary solution exists.

On the other hand, the s-wave uniform state $(u, v, q, \xi) = (u_0, 0, q_0, 0)$, which has been mainly considered in this paper, belongs to the *irregular* solution. The dispersion relations for $\phi \propto e^{ikx}$ become $\lambda^2 = (k^2 - u_0)^2 + q_0^2$, $\omega = k^2 - u_0$. Eliminating k, we find $\lambda^2 - \omega^2 - q_0^2 = 0$, corresponding to Eq. (15), which is easily parametrized as $\lambda = \frac{q_0}{2}(s + s^{-1})$, $\omega = \frac{q_0}{2}(s - s^{-1})$. However, *the wavenumber* $k = \sqrt{\frac{q_0}{2}(s - s^{-1}) + u_0}$ *cannot be expressed as a rational function of the parametrizer* s, thus the BA function ϕ becomes a *double-valued* function on the Riemann surface due to the square root, implying the existence of rank-2 solutions.[93]

Finally, we introduce the concept of *isodispersive phases* to characterize the oscillating region of the tsunami soliton (Fig. 2). For a given reflectionless potential of differential operator $\hat{L}$, we define that the backgrounds of the left and right sides far from the potential ($x \to \pm\infty$) are *isodispersive*. This name is used because both states share the same dispersion relation $\lambda(k)$ via the Jost solution $\phi^{\rm new}(x) = \phi(x) + \int_{-\infty}^{x} K(x, y)\phi(y)dx$. For spatially uniform backgrounds, isodispersive phases are typically connected by trivial gauge transformations; for example, in the integrable spinor Bose condensates with finite density,[94,95] the backgrounds before and after the soliton passes are both polar phases[96,97] at different angles, connected by $U(1) \otimes SO(3)$ group.[98] The same phases with various angles forms the order parameter manifold, whose homotopy groups classify the topological defects.[98,99] On the other hand, the tsunami soliton near the oscillation time $t \simeq 0$ (Fig. 2) shows a reflectionless potential such that the left side is uniform but the right side is oscillating. The multiple tsunami-soliton state (Fig. 3) can support more complicated quasiperiodic backgrounds. In these cases, the set of isodispersive phases may not be compact and physical interpretation of the transformation group is unclear. While the uniform phases are classified by values $\rho_1^{\rm I} = u - 2|\xi|^2$ and $(\rho_3^{\rm I})_{\rm uniform} = |q|^2 + \frac{u^2+v^2}{2} + 2u|\xi|^2 - 2|\xi|^4$, characterizing all quasiperiodic isodispersive phases might require higher-order $\rho_n^{\rm R,I}$'s, remaining to be an open problem.

Summarizing, we have presented the tsunami-soliton and stationary KdV-rock solutions in a classical integrable equation, arising from the BdG operator in parity-mixed SCs. The family of Novikov equations constituting a hierarchy has been generated by Krichever's method. The tsunami solitons provide not only the turning-back dynamics but also the characterization problem of isodispersive quasiperiodic states, which might be used for a new kind of momentum-dependent topological defects and exotic Josephson junctions with transparent scattering properties. The irregular solutions, which are allowed by the non-coprime Lax pair and include the multi KdV rocks, will open up a new application of classical integrable models to physical systems with disordered background.

The unified treatment of the whole hierarchy, the derivation from multicomponent plasmas and fluids by reductive perturbation, and applying to the quadratic-dispersion BdG systems toward full many-body treatment, are all left as future tasks.

The data that support the findings of this article are openly available.[60,81]

1) V. E. Zakharov and A. A. Gelash, Nonlinear stage of modulation instability, Phys. Rev. Lett. **111**, 054101 (2013).
2) N. Akhmediev, Waves that Appear From Nowhere: Complex Rogue Wave Structures and Their Elementary Particles, Frontiers in Physics **8 - 2020** (2021).
3) B. Guo, L. Ling, and Q. P. Liu, Nonlinear Schrödinger equation: Generalized Darboux transformation and rogue wave solutions, Phys. Rev. E **85**, 026607 (2012).
4) E. A. Kuznetsov, Solitons in a parametrically unstable plasma, Sov. Phys. - Dokl. (Engl. Transl.); (United States) **22:9** (1977).
5) T. Kawata and H. Inoue, Inverse Scattering Method for the Nonlinear Evolution Equations under Nonvanishing Conditions, J. Phys. Soc. Jpn. **44**, 1722 (1978).
6) Y.-C. Ma, The Perturbed Plane-Wave Solutions of the Cubic Schrödinger Equation, Stud. Appl. Math. **60**, 43 (1979).
7) N. N. Akhmediev and V. I. Korneev, Modulation instability and periodic solutions of the nonlinear Schrödinger equation, Theoretical and Mathematical Physics **69**, 1089 (1986).
8) J. M. Dudley, G. Genty, F. Dias, B. Kibler, and N. Akhmediev, Modulation instability, Akhmediev Breathers and continuous wave supercontinuum generation, Opt. Express **17**, 21497 (2009).
9) D. H. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, The Journal of the Australian Mathematical Society. Series B. Applied Mathematics **25**, 16 (1983).
10) Y. Ohta and J. Yang, Dynamics of rogue waves in the Davey–Stewartson II equation, J. Phys. A: Math. Theor. **46**, 105202 (2013).
11) D. E. Pelinovsky and R. E. White, Localized structures on librational

and rotational travelling waves in the sine-gordon equation, Proc. R. Soc. A: Math. Phys. Eng. Sci. **476** (2020).
12) G. Mu and Z. Qin, Rogue Waves for the Coupled Schrödinger–Boussinesq Equation and the Coupled Higgs Equation, J. Phys. Soc. Jpn. **81**, 084001 (2012).
13) D. S. Agafontsev and V. E. Zakharov, Integrable turbulence and formation of rogue waves, Nonlinearity **28**, 2791 (2015).
14) M. A. Hoefer, M. J. Ablowitz, I. Coddington, E. A. Cornell, P. Engels, and V. Schweikhard, Dispersive and classical shock waves in Bose-Einstein condensates and gas dynamics, Phys. Rev. A **74**, 023623 (2006).
15) N. N. Bogoliubov, A new method in the theory of superconductivity. I, Sov. Phys. JETP **7**, 41 (1958).
16) P. G. de Gennes, *Superconductivity of metals and alloys*, (Westview Press, Boulder, 1999).
17) D. A. Takahashi, Bogoliubov-de Gennes soliton dynamics in unconventional Fermi superfluids, Phys. Rev. B **93**, 024512 (2016).
18) T. Yoshida, M. Sigrist, and Y. Yanase, Parity-Mixed Superconductivity in Locally Non-centrosymmetric System, J. Phys. Soc. Jpn. **83**, 013703 (2014).
19) T. Mizushima, A. Yamakage, M. Sato, and Y. Tanaka, Dirac-fermion-induced parity mixing in superconducting topological insulators, Phys. Rev. B **90**, 184516 (2014).
20) A. F. Andreev, The thermal conductivity of the intermediate state in superconductors, Sov. Phys. JETP **19**, 1228 (1964).
21) D. A. Takahashi and M. Nitta, On Reflectionless Nature of Self-Consistent Multi-Soliton Solutions in Bogoliubov-de Gennes and Chiral Gross-Neveu Models, J. Low Temp. Phys. **175**, 250 (2014).
22) H. Takayama, Y. R. Lin-Liu, and K. Maki, Continuum model for solitons in polyacetylene, Phys. Rev. B **21**, 2388 (1980).
23) S. A. Brazovskii, S. A. Gordynin, and N. N. Kirova, An exact solution of the Peierls model with an arbitrary number of electrons in the unit cell, JETP Lett. **31**, 456 (1980).
24) S. A. Brazovskii and N. N. Kirova, Excitons, polarons, and bipolarons in conducting polymers, JETP Lett. **33**, 4 (1981).
25) S. A. Brazovskii, N. N. Kirova, and S. I. Matveenko, Peierls effect in conducting polymers, Sov. Phys. JETP **59**, 434 (1984).
26) J. Mertsching and H. J. Fischbeck, The Incommensurate Peierls Phase of the Quasi-One-Dimensional Fröhlich Model with a Nearly Half-Filled Band, Phys. Status Solidi **103**, 783 (1981).
27) Y. Onodera and S. Okuno, Two-Polaron Solution and Its Stability in the Continuum Model of Polyacetylene, J. Phys. Soc. Jpn. **52**, 2478 (1983).
28) S. Okuno and Y. Onodera, Coexistence of a Soliton and a Polaron in Trans-Polyacetylene, J. Phys. Soc. Jpn. **52**, 3495 (1983).
29) S. Brazovskii, I. Dzyaloshinskii, and I. Krichever, Exactly soluble peierls models, Phys. Lett. A **91**, 40 (1982).
30) I. E. Dzyaloshinskii, I. M. Krichever, and J. Chronek, New method of finding dynamic solutions in the Peierls model, Sov. Phys. JETP **68**, 1492 (1988).
31) R. F. Dashen, B. Hasslacher, and A. Neveu, Semiclassical Bound States in an Asymptotically Free Theory, Phys. Rev. D **12**, 2443 (1975).
32) S.-S. Shei, Semiclassical bound states in a model with chiral symmetry, Phys. Rev. D **14**, 535 (1976).
33) J. Feinberg, Marginally stable topologically nontrivial solitons in the Gross-Neveu model, Phys. Lett. B **569**, 204 (2003).
34) K. Machida and H. Nakanishi, Superconductivity under a ferromagnetic molecular field, Phys. Rev. B **30**, 122 (1984).
35) G. Basar and G. V. Dunne, Self-consistent crystalline condensate in chiral Gross-Neveu and Bogoliubov-de Gennes systems, Phys. Rev. Lett. **100**, 200404 (2008).
36) G. Basar and G. V. Dunne, A Twisted Kink Crystal in the Chiral Gross-Neveu model, Phys. Rev. D **78**, 065022 (2008).
37) G. Basar, G. V. Dunne, and M. Thies, Inhomogeneous Condensates in the Thermodynamics of the Chiral NJL(2) model, Phys. Rev. D **79**, 105012 (2009).
38) F. Correa, G. V. Dunne, and M. S. Plyushchay, The Bogoliubov/de Gennes system, the AKNS hierarchy, and nonlinear quantum mechanical supersymmetry, Annals Phys. **324**, 2522 (2009).
39) D. A. Takahashi, S. Tsuchiya, R. Yoshii, and M. Nitta, Fermionic solutions of chiral Gross-Neveu and Bogoliubov-de Gennes systems in nonlinear Schródinger hierarchy, Phys. Lett. B **718**, 632 (2012).
40) D. A. Takahashi and M. Nitta, Self-Consistent Multiple Complex-Kink Solutions in Bogoliubov-de Gennes and Chiral Gross-Neveu Systems, Phys. Rev. Lett. **110**, 131601 (2013).
41) G. V. Dunne and M. Thies, Time-Dependent Hartree-Fock Solution of Gross-Neveu Models: Twisted-Kink Constituents of Baryons and Breathers, Phys. Rev. Lett. **111**, 121602 (2013).
42) G. V. Dunne and M. Thies, Full time-dependent Hartree-Fock solution of large N Gross-Neveu models, Phys. Rev. D **89**, 025008 (2014).
43) M. Thies, Non-Abelian twisted kinks in chiral Gross-Neveu model with isospin, Phys. Rev. D **93**, 085024 (2016).
44) J. Tokimoto, S. Tsuchiya, and T. Nikuni, Excitation of Higgs Mode in Superfluid Fermi Gas in BCS-BEC Crossover, J. Phys. Soc. Jpn. **88**, 023601 (2019).
45) T. Mizushima, M. Ichioka, and K. Machida, Imbalanced Superfluid Phase of a Trapped Fermi Gas in the BCS-BEC Crossover Regime, J. Phys. Soc. Jpn. **76**, 104006 (2007).
46) S. Murakami and M. Wadati, Yangian Symmetry of the δ-Function Fermi Gas, J. Phys. Soc. Jpn. **65**, 1227 (1996).
47) M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models*, (Cambridge University Press, Cambridge, 1999).
48) X.-W. Guan, M. T. Batchelor, and C. Lee, Fermi gases in one dimension: From Bethe ansatz to experiments, Rev. Mod. Phys. **85**, 1633 (2013).
49) J. Y. Lee, X. W. Guan, K. Sakai, and M. T. Batchelor, Thermodynamics, spin-charge separation, and correlation functions of spin-1/2 fermions with repulsive interaction, Phys. Rev. B **85**, 085414 (2012).
50) M. Wadachi and M. Sakagami, Classical Soliton as a Limit of the Quantum Field Theory, J. Phys. Soc. Jpn. **53**, 1933 (1984).
51) M. Wadati, A. Kuniba, and T. Konishi, The Quantum Nonlinear Schrödinger Model; Gelfand-Levitan Equation and Classical Soliton, J. Phys. Soc. Jpn. **54**, 1710 (1985).
52) M. Wadati and A. Kuniba, The Quantum Nonlinear Schrödinger Model; Conserved Quantities, J. Phys. Soc. Jpn. **55**, 76 (1986).
53) J. Sato, R. Kanamoto, E. Kaminishi, and T. Deguchi, Exact Relaxation Dynamics of a Localized Many-Body State in the 1D Bose Gas, Phys. Rev. Lett. **108**, 110401 (2012).
54) J. Sato, R. Kanamoto, E. Kaminishi, and T. Deguchi, Quantum states of dark solitons in the 1D Bose gas, New Journal of Physics **18**, 075008 (2016).
55) Y. Ishiguro, J. Sato, T. Ezaki, and K. Nishinari, Constructing quantum dark solitons with stable scattering properties, Phys. Rev. Res. **4**, L032047 (2022).
56) A. Imambekov and L. I. Glazman, Universal Theory of Nonlinear Luttinger Liquids, Science **323**, 228 (2009).
57) K. A. Matveev and A. Furusaki, Decay of Fermionic Quasiparticles in One-Dimensional Quantum Liquids, Phys. Rev. Lett. **111**, 256401 (2013).
58) I. M. Krichever, Integration of nonlinear equations by the methods of algebraic geometry, Funct. Anal. Its Appl. **11**, 12 (1977).
59) See the supplemental materials.
60) D. Takahashi, Mathematica code for generating Novikov equations (2025), https://doi.org/10.5281/zenodo.17070336.
61) V. G. Makhankov, On stationary solutions of the Schrödinger equation with a self-consistent potential satisfying Boussinesq's equation, Phys. Lett. A **50**, 42 (1974).
62) K. Nishikawa, H. Hojo, K. Mima, and H. Ikezi, Coupled Nonlinear Electron-Plasma and Ion-Acoustic Waves, Phys. Rev. Lett. **33**, 148 (1974).
63) N. Yajima and J. Satsuma, Soliton Solutions in a Diatomic Lattice System, Prog. Theor. Phys. **62**, 370 (1979).
64) Y. Hase and J. Satsuma, An N-Soliton Solution for the Nonlinear Schrödinger Equation Coupled to the Boussinesq Equation, J. Phys. Soc. Jpn. **57**, 679 (1988).
65) V. Makhankov, Dynamics of classical solitons (in non-integrable systems), Phys. Rep. **35**, 1 (1978).
66) I. M. Krichever, Two-dimensional algebraic-geometric operators with self-consistent potentials, Funct. Anal. Its Appl. **28**, 21 (1994).
67) R. G. Scott, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, Dynamics of Dark Solitons in a Trapped Superfluid Fermi Gas, Phys. Rev. Lett. **106**, 185301 (2011).
68) Y. Hidaka, K. Kamikado, T. Kanazawa, and T. Noumi, Phonons, pions, and quasi-long-range order in spatially modulated chiral condensates, Phys. Rev. D **92**, 034003 (2015).
69) Y. Kodama and T. Taniuti, Higher Order Approximation in the Reductive Perturbation Method. I. The Weakly Dispersive System, J. Phys.

Soc. Jpn. **45**, 298 (1978).
70) M. Funakoshi and M. Oikawa, The Resonant Interaction between a Long Internal Gravity Wave and a Surface Gravity Wave Packet, J. Phys. Soc. Jpn. **52**, 1982 (1983).
71) M. Oikawa, M. Okamura, and M. Funakoshi, Two-Dimensional Resonant Interaction between Long and Short Waves, J. Phys. Soc. Jpn. **58**, 4416 (1989).
72) V. E. Zakharov and A. B. Shabat, A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. i, Funct. Anal. Its Appl. **8**, 226 (1974).
73) D. A. Takahashi, On integrable systems and elliptic potentials associated with the quadratic-dispersion Bogoliubov-de Gennes operator, Reports of Institute for Mathematics and Computer Science, Tsuda University **42**, 67 (2021), [written in Japanese], available at: https://doi.org/10.5281/zenodo.17069767.
74) M. Hamanaka and S.-C. Huang, Multi-soliton dynamics of anti-self-dual gauge fields, Journal of High Energy Physics **2022**, 39 (2022).
75) M. Ono, Y. Nishigata, T. Nishio, T. Eguchi, and Y. Hasegawa, Electrostatic Potential Screened by a Two-Dimensional Electron System: A Real-Space Observation by Scanning-Tunneling Spectroscopy, Phys. Rev. Lett. **96**, 016801 (2006).
76) M. Ono, T. Nishio, T. An, T. Eguchi, and Y. Hasegawa, Observation of the screened potential and the Friedel oscillation by low-temperature scanning tunneling microscopy/spectroscopy, Applied Surface Science **256**, 469 (2009).
77) Y. Kodama and H. Yeh, The KP theory and Mach reflection, J. Fluid Mech. **800**, 766 (2016).
78) K. Maruno, D. Suyama, and A. Nagahara, The interactions of dark line solitons in the Davey-Stewartson II system, RIMS kôkyûroku **2109**, 115 (2019).
79) V. B. Matveev, Darboux transformation and explicit solutions of the Kadomtsev-Petviashvili equation, depending on functional parameters, Lett. Math. Phys. **3**, 213 (1979).
80) T. Tsuchida and M. Wadati, The Coupled Modified Korteweg-de Vries Equations, J. Phys. Soc. Jpn. **67**, 1175 (1998).
81) D. Takahashi, Dataset for figure in the paper "Tsunami Solitons Emerging from Superconducting Gap" (2025), https://doi.org/10.5281/zenodo.16936980.
82) T. Miwa, M. Jimbo, and E. Date, *Solitons: Differential Equations, Symmetries, and Infinite Dimensional Algebras*, (Cambridge University Press, Cambridge, 2000).
83) S. Tanaka and E. Date, *KdV houteisiki (The KdV equation)*, (Kinokuniya Shoten, Tokyo, 1979), [written in Japanese].
84) J. L. Burchnall and T. W. Chaundy, Commutative Ordinary Differential Operators, Proc. Lond. Math. Soc. **s2-21**, 420 (1923).
85) J. L. Burchnall and T. W. Chaundy, Commutative Ordinary Differential Operators, Proc. R. Soc. Lond. A. **118**, 557 (1928).
86) I. M. Krichever, Commutative rings of ordinary linear differential operators, Funct. Anal. Its Appl. **12**, 175 (1978).
87) F. Gesztesy and H. Holden, *Soliton Equations and their Algebro-Geometric Solutions*, Cambridge Studies in Advanced Mathematics. (Cambridge University Press, 2003).
88) D. A. Takahashi, Integrable model for density-modulated quantum condensates: Solitons passing through a soliton lattice, Phys. Rev. E **93**, 062224 (2016).
89) B.-F. Feng, L. Ling, and D. A. Takahashi, Multi-breather and high-order rogue waves for the nonlinear Schrödinger equation on the elliptic function background, Stud. Appl. Math. **144**, 46 (2020).
90) B. A. Dubrovin, Theta functions and non-linear equations, Russ. Math. Surv. **36**, 11 (1981).
91) E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, (Springer, Berlin, 1994).
92) I. M. Krichever and S. P. Novikov, Holomorphic bundles over Riemann surfaces and the Kadomtsev–Petviashvili equation. I, Funct. Anal. Its Appl. **12**, 276 (1979).
93) I. M. Krichever and S. P. Novikov, Holomorphic bundles over algebraic curves and non-linear equations, Russ. Math. Surv. **35**, 53 (1980).
94) J. Ieda, M. Uchiyama, and M. Wadati, Inverse scattering method for square matrix nonlinear Schrödinger equation under nonvanishing boundary conditions, J. Math. Phys. **48**, 013507 (2007).
95) M. Uchiyama, J. Ieda, and M. Wadati, Dark Solitons in $F = 1$ Spinor Bose-Einstein Condensate, J. Phys. Soc. Jpn. **75**, 064002 (2006).
96) T.-L. Ho, Spinor Bose Condensates in Optical Traps, Phys. Rev. Lett. **81**, 742 (1998).
97) T. Ohmi and K. Machida, Bose-Einstein Condensation with Internal Degrees of Freedom in Alkali Atom Gases, J. Phys. Soc. Jpn. **67**, 1822 (1998).
98) Y. Kawaguchi, M. Kobayashi, M. Nitta, and M. Ueda, Topological Excitations in Spinor Bose-Einstein Condensates, Prog. Theor. Phys. Suppl. **186**, 455 (2010).
99) M. Nakahara, *Geometry, Topology, and Physics*, (IOP Publishing, Bristol, England, 2nd edition, 2003).