# On a simple dynamical map with a flooring function

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**Abstract.** We report here a computational study of a recently proposed simple dynamical map. Despite its simplicity, it offers a wide variety of dynamical paths depending on the initial value. Some analytical properties and future directions are also discussed.

**Keywords:** Flooring function, Non-linear dynamics, Dynamical map

## 1. Introduction

Investigations of dynamical maps have gained much attention in mathematical sciences. A representative example is a logistic map[1, 2] that has been investigated extensively. Many researchers are intrigued by the chaotic dynamics produced by this very simple map.

Against this background, we numerically study here a recently proposed dynamical map, which we call the "N-map". Despite its simplicity, this map gives rise to a variety of dynamical patterns; monotonic approach to a single fixed point, divergence, and oscillations. Interestingly, the map does not contain any tuning (bifurcation) parameter and all of these can be achieved solely by the change of the initial value. Some analytical properties are also discussed as well as a future extension of the study of the N-map.

## 2. N-map

The N-map is given by the following first-order recursion relation.

$$X_{n+1} = 3F[X_n + \frac{1}{2}] - 2X_n, \quad X_0 = a, \quad (0 < a < 1, \quad n = 0, 1, 2, \dots). \tag{1}$$

where $F[x]$ is a flooring function which returns largest integer $N$ with $N \leq x$.

### *2.1. Analysis*

We can analytically prove some properties of this map, particularly concerting oscillatory behaviors.

### *2.2. Condition for periodic dynamics*

**Lemma 2.1** *If $X_{n+k} = X_n$ for some $n, k \in \mathbb{N}_{\geq 1}$, $a$ is a rational number.*

**Proof**

For any $m$, we have $X_{m+1} + 2X_m \in \mathbb{Z}$. Thus, for any $n, k \in \mathbb{N}_{\geq 1}$, $X_{n+k} - (-2)^k X_n \in \mathbb{Z}$. Therefore, if $X_{n+k} = X_n = c$, $c \in \frac{1}{1-(-2)^k}\mathbb{Z} \subseteq \mathbb{Q}$. As $a = \frac{s+(-1)^{n-1}c}{2^{n-1}}$ with some integer $s$, it follows that $a \in \mathbb{Q}$. □

This lemma tells us the following:

(i) If the dynamical path is periodic, $a$ must be a rational number. So, when the initial value $a$ is an irrational number, then the path cannot be periodic.

(ii) On the other hand, we note that we can have a non-periodic path for a certain rational number. It can be easily shown that

- If $a = \frac{1}{3}$, then $X_\infty \to -\infty$,
- If $a = \frac{2}{3}$, then $X_\infty \to \infty$.

*2.3. Period of periodic dynamics*

Let us further analytically investigate the period when a periodic solution arises. In the following, we assume $a$ $(0 \leq a < 1)$ is a rational number up to $m$ decimal points, i.e.,

$$\{m \in \mathbb{N}_{\geq 0} \mid 10^m a \in \mathbb{Z}\} \neq \emptyset$$

**Definition 2.2** *For such $a$, we define the followings.*

*(i)* $m(a) = \min\{m \in \mathbb{N}_{\geq 0} \mid 10^m a \in \mathbb{Z}\}$.

*(ii)* $\ell(a) = \max\{\ell \in \mathbb{N}_{\geq 0} \mid \frac{10^{m(a)}a}{5^\ell} \in \mathbb{Z}\}$.

*(iii)* $q(a) = \min\{m(2^n a) \mid n \in \mathbb{N}_{\geq 0}\}$.

These definitions mean respectively:

(i) $a$ is a rational number with $m(a)$ decimal points.

(ii) The integer, $10^{m(a)}$, (that is the decimal part of $a$) can be divisible by 5 for $\ell(a)$ times.

(iii) When $n$ is large enough, $X_n$has $q(a)$ decimal points.

**Remark 2.3** *The relation $q(a) = m(a) - \ell(a)$ holds.*

**Example 2.4**

(A) $m(0.04) = 2$, $\ell(0.04) = 0$, $q(0.04) = 2$.

(B) $m(0.05) = 2$, $\ell(0.05) = 1$, $q(0.05) = 1$.

(F) $m(0.5) = 1$, $\ell(0.5) = 1$, $q(0.5) = 0$.

When $q(a) = 0$, we have $a_n \in \mathbb{Z}$ for some$n \in \mathbb{N}$ and it stays as constant. Thus, we consider the case of $q(a) > 0$ in the following.

**Lemma 2.5** *Consider the case $q(a) > 0$. If $a + b = r$ for some $r \in \mathbb{Z}$, then $f(a) + f(b) = r$.*

**Proof**

It follows from the definition of $f$. □

We now prepare the following two general lemmas.

**Lemma 2.6** *When the remainder of dividing $c \in \mathbb{Z}$ by 5 is 4, the followings hold*

*(i)* $1 + c^3 + c^4$ *is a multiple of* 5*. Also, the remainder of* $c^5$ *divided by* 5 *is* 4

*(ii) If* $1 + c$ *is a multiple of* $5^k$ *for* $k \in \mathbb{N}_{>0}$*, then* $1 + c^5$ *is a multiple of* $5^{k+1}$.

**Proof**

(i) In $\mathbb{Z}/5\mathbb{Z}$, $c = -1$. Hence, $1 - c + c^2 - c^3 + c^4 = 5 = 0$ and $c^5 = -1$.

(ii) As $(1 + c^5) = (1 + c)(1 - c + c^2 - c^3 + c^4)$, the result follows immediately from (i). $\square$

**Proposition 2.7** *For any $q \in \mathbb{N}_{>0}$, $(1 + 2^{2 \cdot 5^{q-1}}) = (1 + 4^{5^{q-1}})$ is divisible by $5^q$.*

**Proof**

When $q = 1$, it is obvious as $1 + 2^{2 \cdot 5^0} = 5$. The rest follows from lemma 2.6 by induction on $q$. $\square$

With these preparations, we are now in the position of providing half period $p(a)$ of periodic dynamics of $\{X_n\}$.

**Definition 2.8** *For $q(a) > 0$, we set*

$$p(a) = \min\{p \in \mathbb{N}_{\geq 0} \mid 5^{q(a)} divides (1 + (-2)^p)\}$$

.

We note that this set on the righthand side is not empty, as it has $2 \cdot 5^{q(a)-1}$ as an element by Proposition 2.7.

**Remark 2.9** *As $(1 + (-2)^{p(a)})$ is divisible by $5q(a)$, $p(a) \equiv 2 \mod 4$. In particular $p(a)$ is an even number.*

For example, we have $p(a) = 2$ for $q(a) = 1$, and $p(a) = 10$ for $q(a) = 2$. Even though the details needs to be verified, we expect in general, $p(a) = 2 \cdot 5^{q(a)-1}$, so that for $q(a) = 3, 4, 5, \ldots$, $p(a) = 50, 250, 1250, \ldots$.

We show in the following that $p(a)$ is the half period when the dynamical paths $X_n$ is periodic. For that purpose, we focus on the periodicity of the decimal part. We define $\langle r \rangle = r - [r]$ as the decimal part of $r \in \mathbb{R}$.

**Proposition 2.10** *Let $q(a) > 0$, then for some $N \in \mathbb{N}_{\geq 0}$, $\langle X_{N+p(a)} \rangle + \langle X_N \rangle = 1$ holds. Also for this $N$, the followings are ture.*

*(i) If we define $u(a) \equiv X_{N+p(a)} + X_N$, then $u(a) \in \mathbb{Z}$.*

*(ii) For any $n \geq N$, $X_{n+p(a)} + X_n = u(a)$.*

*(iii) For any $n \geq N$, $X_{n+2p(a)} = X_n$.*

**Proof**

For $n \in \mathbb{N}_{\geq 1}$, $\langle X_n \rangle - (-2)^{n-1}\langle a \rangle \in \mathbb{Z}$. This leads to $(\langle X_{n+p(a)} \rangle + \langle X_n \rangle) - (-2)^{n-1}(1 + (-2)^{p(a)})\langle a \rangle \in \mathbb{Z}$. Therefore, if we take $N$ large enough, we obtain $\langle X_{N+p(a)} \rangle + \langle X_N \rangle \in \mathbb{Z}$ from the way we defined $p(a)$. As $0 < \langle X_{N+p(a)} \rangle, \langle X_N \rangle < 1$, $\langle X_{N+p(a)} \rangle + \langle X_N \rangle = 1$.

(i) $X_{N+p(a)} + X_N = [X_{N+p(a)}] + [X_N] + \langle X_{N+p(a)} \rangle + \langle X_N \rangle \in \mathbb{Z}$.

(ii) It follows from Lemma 2.5.

(iii) From (3), $X_{n+2p(a)} = u(a) - X_{n+p(a)} = u(a) - (u(a) - X_n) = X_n$. $\square$

**Corollary 2.11** *Let $q(a) > 0$. For $b = 1 - a$, we define $Y_n = f_{(n-1)}(b)$ $(n \in \mathbb{N}_{\geq 1})$, i.e., $\{Y_n\}$ is the dynamical path with the initial value $b$. Then, the following holds.*

*(i) For any $n \in \mathbb{N}_{\geq 1}$, $X_n + Y_n = 1$.*

*(ii) For any $n \geq N$, $Y_n = (1 - u(a)) + X_{n+p(a)}$. Here, $N$ is set as large enough given $a$.*

**Proof**

(i) It follows from Lemma 2.5.

(ii) As $X_{n+p(a)} = u(a) - X_n$, $Y_n = 1 - X_n = 1 - (u(a) - X_{n+p(a)}) = (1 - u(a)) + X_{n+p(a)}$. $\square$

*2.4. Numerical Simulations*

In order to verify and gain further insight, we performed numerical simulation for this map with varying the initial value $a$. In Fig.1, we have shown samples of periodic dynamics with periods 4 (A,C) and 20 (B, D) verifying Proposition 2.10. Also, in Fig.2, we give examples of the case described in Corollary 2.11. The representative results of non-periodic cases are shown in Fig. 3, confirming such properties as the divergences for $a = 1/3$ and $a = 1/3$. Together, these representative paths show a variety of dynamical patterns out of this simple map just by the change of initial value $a$: monotonic convergence and divergence, non-monotonic convergence and divergence, oscillations, and complex oscillatory path, complex non-periodic path.

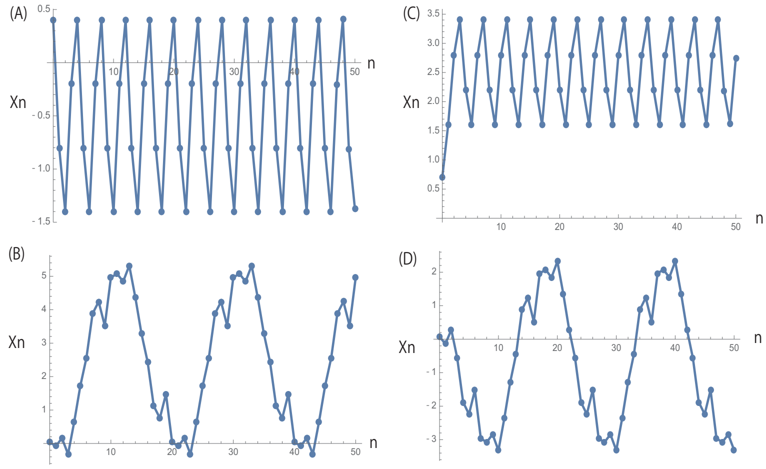


**Figure 1.** Samples of periodic dynamical paths from the N-map. The values of $a$ are (A) 0.4, (B) 0.04, (C) 0.7, (D) 0.07.

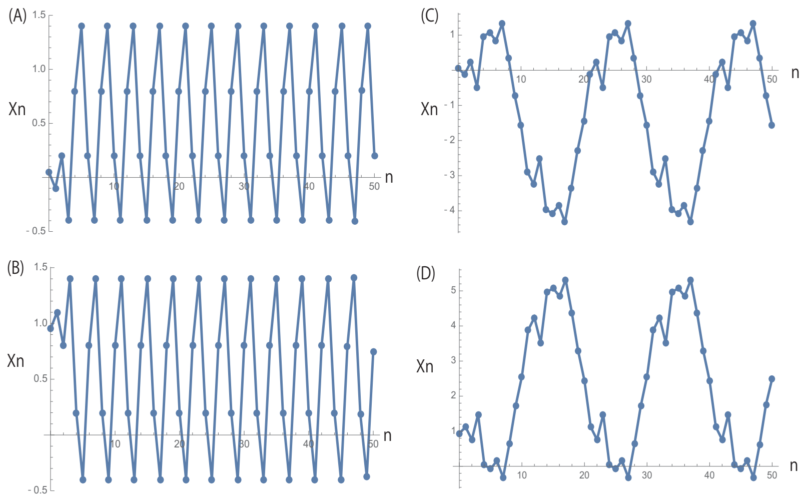


**Figure 2.** Samples of dynamical paths from the N-map for $a$ and $1 - a$. The values of $a$ are (A) 0.05, (B) 0.95, (C) 0.06, (D) 0.94

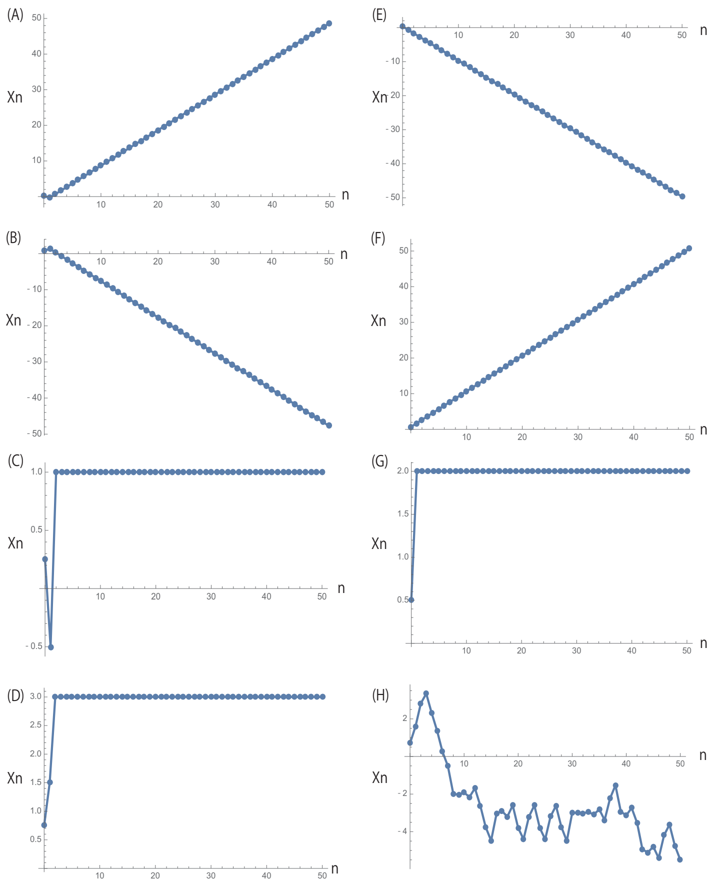

**Figure 3.** Samples of non-periodic dynamical paths from the N-map. The values of $a$ are (A) 1/6, (B) 5/6, (C) 1/4, (D) 3/4, (E) 1/3, (F) 2/3, (G) 1/2, (H) $1/\sqrt{2}$.

## 3. Discussion

We have presented a rather preliminary numerical study of a simple map with a flooring function. More thorough investigations are needed to reveal the nature of this N-map. The natural extension of the N-map is given as

$$X_{n+1} = \alpha F[X_n + \frac{1}{2}] - \beta X_n, \quad X_0 = a, \quad (0 < a < 1, \quad n = 0, 1, 2, \dots), \tag{2}$$

With the real parameters $\alpha$ and $\beta$. We expect to see more intricate dynamics for some ranges of these parameters.

**Acknowledgments and Declaration**

The author would like to thank Prof. Hiroyuki Nakaoka of Nagoya University for his enlightening comments. This work was supported by funding from Ohagi Hospital, Hashimoto, Wakayama, Japan, and by Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science No.19H01201.Author has no conflicts of interest to declare.

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[3] This map first appeared in the entrance examination of Nagoya University (Japan) on February 26, 2021.