Electric potentials and field lines for uniformly-charged tube and cylinder expressed by Appell's hypergeometric function and integration of $Z(u|m)\mathrm{sc}(u|m)$

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The closed-form expressions of electric potentials and field lines for a uniformly-charged tube and cylinder are presented using elliptic integrals and Appell's hypergeometric functions, where field lines are depicted by introducing the concept of the field line potential in axisymmetric systems, whose contour lines represent electric field lines outside the charged region, thought of as an analog of the conjugate harmonic function in the presence of non-uniform metric. The field line potential for the tube shows a multi-valued behavior and enables us to define a topological charge. The integral of $Z(u|m)\,\mathrm{sc}(u|m)$, where Z and sc are the Jacobi zeta and elliptic functions, is also expressed by Appell's hypergeometric function as a by-product, which was missing in classical tables of formulas.

Introduction — Multi-variable generalizations of hypergeometric functions such as Appell's hypergeometric functions,[1] the Kampé de Fériet functions, and the Lauricella functions,[2] emerge in modern physics in a variety of ways; the example includes the Feynman integrals,[3,4] the capacity of entanglement,[5] and the loop amplitude in photon-photon scattering.[4] Those on a finite field have also been studied in number theory.[6] We should also mention the recent developments on the elliptic generalization in mathematical physics.[7] It is not of academic interest that the solutions of physical problems can be written by these special functions, because their linear and higher-order transformation formulae, integral representations, and differential equations enable us to predict their global behaviors beyond the definition series which only has a finite radius of convergence.

Here we report an application of Appell's hypergeometric function to rather an elementary problem. That is, we provide closed-form expressions for electric potentials and field lines for a uniformly-charged tube and cylinder. In the integration of these problems, after separating the terms expressible by elementary and elliptic integrals, only the following term remains:

$$I_{\mathrm{hyg}}(m, A; \theta) := \int_0^\theta d\theta \tanh^{-1} \frac{A}{\sqrt{1 - m \sin^2 \frac{\theta}{2}}}. \qquad (1)$$

This integral does not reduce to an abelian integral by any change of variable, so it cannot be expressed by the Riemann theta functions which are used to express finite-zone solutions in classical integrable systems.[8,9] However, since the parameter derivatives $\frac{\partial I_{\mathrm{hyg}}}{\partial A}$ and $\frac{\partial I_{\mathrm{hyg}}}{\partial m}$ are written by elliptic integrals, I_{hyg} allows a double integral expression of the algebraic function, implying that it could possibly be expressed by some multi-variable hypergeometric functions. Indeed, using Appell's hypergeometric function of the second kind[1]

$$F_2^{\mathrm{Appell}}\left({\alpha;\beta,\beta' \atop \gamma,\gamma'} ; x, y \right) := \sum_{j,l=0}^{\infty} \frac{(\alpha)_{j+l}(\beta)_j(\beta')_l}{j!\,l!\,(\gamma)_j(\gamma')_l} x^j y^l, \qquad (2)$$

where $(x)_n = x(x+1)\dots(x+n-1)$ is the Pochhammer symbol, the above integral, writing $s = \sin\frac{\theta}{2}$, is given by

$$I_{\mathrm{hyg}}(m, A; \theta) = \pi A \,\mathrm{sgn}(s) F_2^{\mathrm{Appell}}\left({\frac12;\frac12,1 \atop 1,\frac32} ; m, A^2 \right) - 2As\sqrt{1-s^2} \sum_{k=0}^{\infty} \frac{(1)_k}{(\frac32)_k}(1-s^2)^k F_2^{\mathrm{Appell}}\left({\frac12;1+k,1 \atop 1,\frac32} ; ms^2, A^2 \right). \qquad (3)$$

In particular, the definite integral $I_{\mathrm{hyg}}(m, A; \pi)$ is expressed only by the first term and hence the above-mentioned problem can be solved.

Furthermore, rewriting the elliptic integral of the third kind using the Jacobi zeta and theta functions,[10,11] we provide an integral of $Z(u|m)\,\mathrm{sc}(u|m)$, which was missing in classical table of formulas for elliptic functions.[12,13] Thus, even a fundamental problem in classical electromagnetism could sometimes offer an opportunity to improve our knowledge on higher transcendental functions.

The paper is organized as follows. First, we introduce the concept of the field line potential in axisymmetric systems whose contour line describes the electric field line. We in particular emphasize that its definition is restricted to the chargeless regions. Next, we provide and summarize the exact expressions of electric potentials and field line potentials for a uniformly-charged cylinder and tube. We also point out that, for the case of tube, the field line potential is multi-valued and has a topological charge. Lastly, we prove Eq. (3), and also provide a by-product integration formula for a product of the Jacobi zeta and elliptic functions, which was missing in classical tables of formulas for elliptic functions.[12,13]

Throughout the paper, we follow the notations of elliptic integrals and functions in Ref. 13. So, the incomplete elliptic integrals of the first, the second, and the third kind are denoted by $F(\varphi|m)$, $E(\varphi|m)$, and $\Pi(n;\varphi|m)$.

Field line potential — Before going to the main subject of this paper, we use a few paragraphs to introduce the field line potential whose equipotential lines coincide with the electric field lines. We write the cylindrical coordinate as (r, θ, z) and consider axisymmetric charge distribution $\rho(r, \theta, z) = \rho(r, z)$. With the unit choice $4\pi\epsilon_0 = 1$, the electric potential at position

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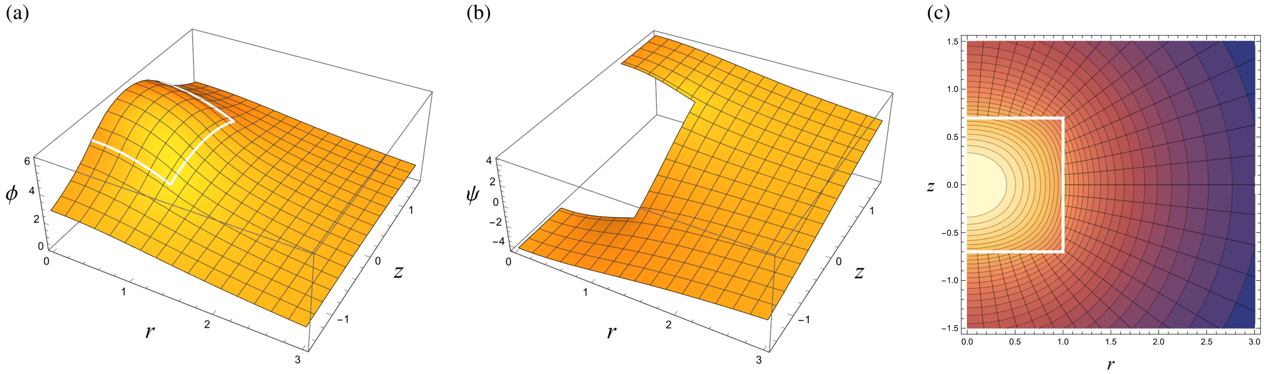

Fig. 1. The electric potential and field line potential made by a uniformly charged cylinder. Figures (a) and (b) represent $\phi^{\rm cyl}$ and $\psi^{\rm cyl}$ with parameters $\rho_0 = 1$, $R = 1$, and $Z = 0.7$. The potential $\phi^{\rm cyl}$ is C^1 on the surface. Figure (c) is a contour plot, showing that equipotential lines of ϕ and ψ are orthogonal. Inside the cylinder, ψ is not given by Eq. (5), so we leave it unplotted, though the field lines themselves do exist.

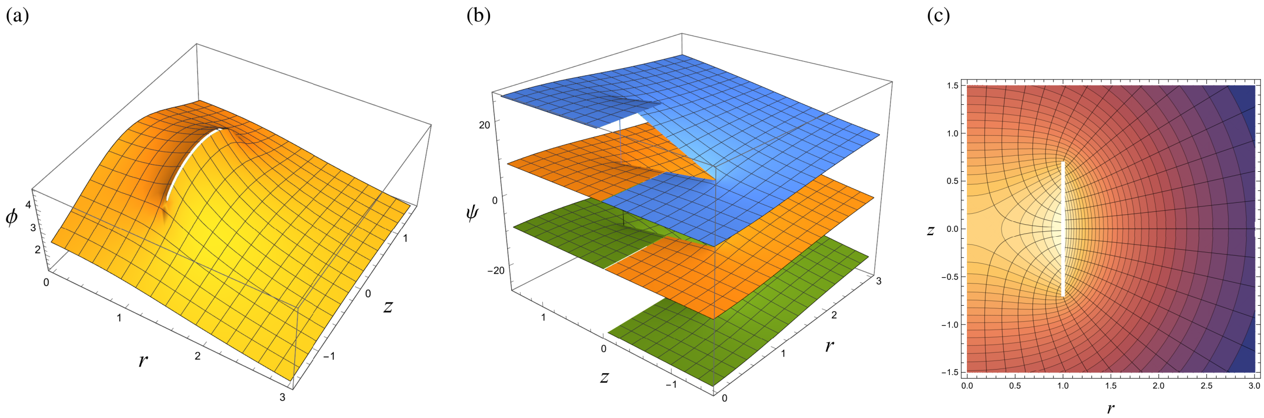

Fig. 2. The electric potential and field line potential made by a uniformly charged tube. We set parameters $\sigma_0 = 1$, $R = 1$, and $Z = 0.7$. Figure (a) represent $\phi^{\rm tube}$, which has a non-differentiable corner at $r = R$, $|z| < Z$. In Figure (b), we plot $\psi^{\rm tube} + n\Delta\psi^{\rm tube}$ with $n = 0, \pm 1$, showing the log-like multi-valued character. Figure (c) is a contour plot.

$\boldsymbol{r} = (r\cos\theta, r\sin\theta, z)$ is given by

$$\phi(r,z) = \iiint \frac{\rho(r',z')r'dr'd\theta'dz'}{L}, \quad L = |\boldsymbol{r}' - \boldsymbol{r}|. \tag{4}$$

Then, the field line potential ψ whose contour lines represent the electric field lines *defined only outside the charged region* is given by

$$\psi(r,z) = \iiint \frac{r(z-z')(r + r'\cos\theta')}{L[L^2 - (z'-z)^2]}\rho(r',z')r'dr'd\theta'dz' + \text{(correction originating from a uniform field)}, \tag{5}$$

if $\rho(r,z) = 0$. Below we derive Eq. (5), and explain why this ψ is valid only for the chargeless region.

While the electric field line is originally defined by a streamline of the electric field, i.e., the solution of the ordinary differential equation (ODE) $\frac{d}{ds}\boldsymbol{x}(s) = \boldsymbol{E}(\boldsymbol{x}(s))$, we follow another formulation. Let us consider a coordinate transformation $u = \phi(r,z)$ and $v = \psi(r,z)$ with θ fixed. Since the equipotential line of the electric potential ϕ and the field lines are everywhere orthogonal, the desired function ψ must be chosen so that the off-diagonal element of the metric tensor in the new coordinate vanishes: $g_{uv} = 0 \leftrightarrow \phi_r\psi_r + \phi_z\psi_z = 0$. Let us assume the form $\psi_r = f(r)\phi_z$, $\psi_z = -f(r)\phi_r$. Using the Laplace equation outside the charged region $\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0$ and the compatibility $\phi_{rz} = \phi_{zr}$, we conclude that $f(r) = r$, that is,

$$\psi_r = r\phi_z, \quad \psi_z = -r\phi_r, \tag{6}$$

and ψ satisfies the partial differential equation (PDE) given later [Eq. (8)]. Integrating the latter of Eq. (6), we find $\psi = -\int dz r\frac{\partial\phi}{\partial r}$, which yields Eq. (5), where without losing generality we set $\theta = \pi$ by axisymmetry. The second term of Eq. (5) complements the lost information on a uniform electric field, since the formula is obtained by an integration of the derivative ϕ_r. Note that the equivalence of the ODE-based streamline formulation and this PDE-based method can be traced back to the method of characteristic curves.[14)]

Let us write the electric field $\boldsymbol{E} = -\nabla\phi$ and its conjugate field $\boldsymbol{\mathcal{B}} = \nabla\psi$. The field $\boldsymbol{\mathcal{B}}$ has a mathematical property similar to magnetic fields, but not a real magnetic field. Taking into account the componentwise relation of Eq. (6), $E_r = \frac{1}{r}\mathcal{B}_z$, $-E_z = \frac{1}{r}\mathcal{B}_r$, their fundamental laws are given by

$$\nabla\cdot\boldsymbol{E} = \frac{(\nabla\times\boldsymbol{\mathcal{B}})_\theta}{r} = -(\phi_{rr} + \tfrac{1}{r}\phi_r + \phi_{zz}) = 4\pi\rho, \tag{7}$$

$$(\nabla\times\boldsymbol{E})_\theta = \nabla\cdot\left(\frac{\boldsymbol{\mathcal{B}}}{r}\right) = \psi_{rr} - \tfrac{1}{r}\psi_r + \psi_{zz} = 0. \tag{8}$$

Equation (7) explains why we cannot use Eq. (5) inside a charged region $\rho \neq 0$ — the potential function is introduced only for a rotation-free vector field.

Here we give further remarks (i)-(v) about ψ. (i) ψ makes sense only for axisymmetric systems. For non-symmetric ρ, the path of each field line is not closed in one plane, and hence no natural choice for remaining two coordinates perpendicular to ϕ exists. (ii) ψ is possibly a multivalued function, though its gradient $\nabla\psi$ is always a single-valued vector field. (iii) The concept of the electric field line itself does survive even inside the charged region $\rho \neq 0$; what we only claim here is that the function satisfying the relation $\phi_r\psi_r + \phi_z\psi_z = 0$ cannot be given by Eq. (5). (iv) ψ satisfies the same superposition principle as ϕ; that is, if ψ_i, $i = 1, 2$, are the solution for the axisymmetric charge profile ρ_i, then $\psi_1 + \psi_2$ is the solution for $\rho_1 + \rho_2$. For example, ψ for a point charge q at origin is easily found to be $\psi = \frac{qz}{\sqrt{r^2+z^2}}$, and hence ψ for two point charges $q_\pm$ located at $(r, z) = (0, \pm a)$ is given by their superposition: $\psi = \sum_\pm \frac{q_\pm(z\mp a)}{\sqrt{r^2+(z\mp a)^2}}$. Drawing the equipotential line of this ψ, we soon find a familiar picture of the electric field lines created by two point charges found in textbooks. (v) In the two-dimensional electromagnetism, the field line potential ψ is just a conjugate harmonic function of ϕ and they satisfy the Cauchy-Riemann relation, a counterpart of Eq. (6); the example of single point charge is $\phi = -\log r = -\ln\sqrt{x^2+y^2}$ and $\psi = \theta = \tan^{-1}\frac{y}{x}$ and the latter is indeed multi-valued. The present formulation for axisymmetric three-dimensional systems can therefore be regarded as a generalization with non-uniform metric.

Cylinder — Now let us go to the uniformly-charged cylinder. Writing $L = \sqrt{r^2 + r_0^2 + 2rr_0\cos\theta + z^2}$ and $L_0 = [L]_{\theta=0} = \sqrt{(r+r_0)^2 + z^2}$, the triple indefinite integrals

$$I^{\rm cyl}(r,\theta,z;r_0) = \iiint \frac{rdrd\theta dz}{L}, \tag{9}$$

$$J^{\rm cyl}(r,\theta,z;r_0) = \iiint \frac{-r_0z(r_0 + r\cos\theta)rdrd\theta dz}{L(L^2-z^2)} \tag{10}$$

are given by $I^{\rm cyl}(r,\theta,z;r_0) = \sum_{\rm T=trig,ell,hyg} I^{\rm cyl}_{\rm T}(r,\theta,z;r_0)$ and $J^{\rm cyl}(r,\theta,z;r_0) = \sum_{\rm T=trig,ell} J^{\rm cyl}_{\rm T}(r,\theta,z;r_0)$, where

$$I^{\rm cyl}_{\rm hyg} = \frac{r^2}{2} I_{\rm hyg}(m, A; \theta), \tag{11}$$

$$I^{\rm cyl}_{\rm trig} = \frac{-r_0^2\sin2\theta}{4}\tanh^{-1}\frac{z}{L} - zr_0\sin\theta\tanh^{-1}\frac{r+r_0\cos\theta}{L} + \frac{r_0^2\cos2\theta}{4}\tan^{-1}\frac{Lr_0\sin\theta}{z(r+r_0\cos\theta)}, \tag{12}$$

$$I^{\rm cyl}_{\rm ell} = \frac{-3z(r_0^2+z^2)}{4L_0}F(\tfrac{\theta}{2}|m) + \frac{3zL_0}{4}E(\tfrac{\theta}{2}|m) + \frac{zr^2(r-r_0)}{4L_0(r+r_0)}\Pi\Big(\frac{4rr_0}{(r+r_0)^2};\frac{\theta}{2}\Big|m\Big) + \frac{z(2z^2-r_0^2)}{4L_0}\sum_{\alpha=\pm}\Big[1-\frac{n_\alpha}{2}\Big(1+\frac{r}{r_0}\Big)\Big]\Pi(n_\alpha;\tfrac{\theta}{2}|m), \tag{13}$$

$$J^{\rm cyl}_{\rm trig} = \frac{(3r_0^3-4r_0z^2)\sin\theta - r_0^3\sin3\theta}{8}\tanh^{-1}\frac{r+r_0\cos\theta}{L} - \frac{r_0^2z\sin2\theta}{2}\tanh^{-1}\frac{z}{L} + \frac{r_0^2\cos2\theta}{2}\tan^{-1}\frac{Lr_0\sin\theta}{z(r+r_0\cos\theta)} + \frac{Lr_0\sin\theta(-r+3r_0\cos\theta)}{6}, \tag{14}$$

$$J^{\rm cyl}_{\rm ell} = \frac{L_0[z^2-2(r^2+r_0^2)]}{6}E(\tfrac{\theta}{2}|m) + \frac{2(r^2-r_0^2)^2+z^2(r_0^2-2r^2-z^2)}{6L_0}F(\tfrac{\theta}{2}|m) + \frac{z^2r^2(r-r_0)}{2L_0(r+r_0)}\Pi\Big(\frac{4rr_0}{(r+r_0)^2};\frac{\theta}{2}\Big|m\Big) - \frac{r_0^2z^2}{2L_0}\sum_{\alpha=\pm1}\Big[1-\frac{n_\alpha}{2}\Big(1+\frac{r}{r_0}\Big)\Big]\Pi(n_\alpha;\tfrac{\theta}{2}|m). \tag{15}$$

where $m = \frac{4rr_0}{L_0^2}$, $A = \frac{z}{L_0}$, $n_\pm = \frac{2r_0}{r_0\pm\sqrt{r_0^2+z^2}}$, and $I_{\rm hyg}$ is given by (1) or (3). We bequeath another expression for the last terms in Eqs. (13) and (15): $1-\frac{n_\alpha}{2}\Big(1+\frac{r}{r_0}\Big) = \frac{L_0}{2r_0}s_\alpha\sqrt{n_\alpha(n_\alpha-m)}$ with $s_\pm = {\rm sgn}\,(\sqrt{r_0^2+z^2}\mp r)$, which plays a key role in determination of $\phi^{\rm cyl}_{\rm corr}$ and $\psi^{\rm cyl}_{\rm corr}$ (see below) using the formula 117.02 in Ref. 12.

The electric potential $\phi^{\rm cyl}(r,z)$, made by a cylinder of radius R and height $2Z$ with uniform charge density ρ_0, is then constructed as follows. If the above indefinite integral formula had no artificial singularity, it would be $\phi = \rho_0\Big[[2I^{\rm cyl}(r',\pi,z'-z;r)]_{r'=0}^{r'=R}\Big]_{z'=-Z}^{z'=Z}$. However, we actually must include the contributions from the branch shift of the multivalued functions $\tan^{-1}$ and Π. The same also holds for the field line potential $\psi^{\rm cyl}(r,z)$, but it has one more correction shown in Eq. (5). After careful determination of these corrections, we get $\phi^{\rm cyl} = \phi^{\rm cyl}_{\rm hyg} + \phi^{\rm cyl}_{\rm ell} + \phi^{\rm cyl}_{\rm corr}$ and $\psi^{\rm cyl} = \psi^{\rm cyl}_{\rm ell} + \psi^{\rm cyl}_{\rm corr}$, where

$$\phi^{\rm cyl}_{\rm T} = \rho_0\sum_{\beta=\pm1}2\beta I^{\rm cyl}_{\rm T}(R,\pi,\beta Z-z;r), \quad {\rm T = hyg, ell}, \tag{16}$$

$$\phi^{\rm cyl}_{\rm corr} = \pi\rho_0\Big[r^2H(r-R) - 2(z^2+Z^2)\Big]H(Z-|z|) - 4\pi\rho_0|z|H(|z|-Z), \tag{17}$$

$$\psi^{\rm cyl}_{\rm ell} = \rho_0\sum_{\beta=\pm1}2\beta J^{\rm cyl}_{\rm ell}(R,\pi,\beta Z-z;r), \tag{18}$$

$$\psi^{\rm cyl}_{\rm corr} = -2\pi\rho_0r^2zH(Z-|z|) + 2\pi\rho_0Z\,{\rm sgn}\,z\Big[-r^2+R^2H(R-r)\Big]H(|z|-Z), \tag{19}$$

and $H(x)$ is the Heaviside function. We can check $\phi^{\rm cyl} \to \frac{Q}{\sqrt{r^2+z^2}}$ and $\psi^{\rm cyl} \to \frac{Qz}{\sqrt{r^2+z^2}}$ with total charge $Q = 2\pi R^2Z\rho_0$ at spatial infinity $\sqrt{r^2+z^2} \to \infty$. The plots are shown in Fig. 1.

Tube — Next, we consider the tube whose thickness is negligible. The double indefinite integral formulae are given by

$$I^{\rm tube}(r,\theta,z;r_0) = \iint\frac{d\theta dz}{L} = I_{\rm hyg}(m,A;\theta), \tag{20}$$

$$J^{\rm tube}(r,\theta,z;r_0) = \iint\frac{-r_0z(r_0+r\cos\theta)d\theta dz}{L(L^2-z^2)} = \frac{r^2-r_0^2}{L_0}F(\tfrac{\theta}{2}|m) - L_0E(\tfrac{\theta}{2}|m) + \frac{z^2}{L_0}\frac{r-r_0}{r+r_0}\Pi\Big(\frac{4rr_0}{(r+r_0)^2};\frac{\theta}{2}\Big|m\Big), \tag{21}$$

where the definitions of L, L_0, m, A are the same as those of the cylinder.

Determining the correction terms in the same way as the case of cylinder, the electric (field line) potentials $\phi^{\rm tube}(r,z)$ and $\psi^{\rm tube}(r,z)$ made by a tube of radius R and height $2Z$ with uniform surface charge density σ_0 is given by

$$\phi^{\rm tube} = \sigma_0R\sum_{\beta=\pm1}2\beta I^{\rm tube}(R,\pi,\beta Z-z;r), \tag{22}$$

$$\psi^{\rm tube} = \sigma_0R\sum_{\beta=\pm1}2\beta J^{\rm tube}(R,\pi,\beta Z-z;r) + \psi^{\rm tube}_{\rm corr}, \tag{23}$$

where

$$\psi^{\rm tube}_{\rm corr} = 4\pi\sigma_0RZ\,{\rm sgn}\,z\,H(R-r). \tag{24}$$

The plots are shown in Fig. 2. Since the region with $\rho = 0$ where ψ is definable is not simply connected, we can observe its topological character. That is, ψ is a multivalued function possessing the log-like branch structure, and the jump value between the neighboring branches

$$\Delta\psi^{\text{tube}} = 8\pi RZ\sigma_0 = 2Q \tag{25}$$

can be thought of as a topological charge, where Q is a total electric charge of the tube. Thus, $\psi^{\text{tube}} + n\Delta\psi^{\text{tube}}$ with $n \in \mathbb{Z}$ becomes a single-valued function on the extended plane made by cut-and-glue of right half planes [Fig. 2 (b)]. Accidentally, the topological and electric charges have the same dimension now. Indeed, if $4\pi\epsilon_0 = 1$, the electric potential has the dimension $[\phi] = \left[\int \frac{\rho dr}{|\mathbf{r}-\mathbf{r}'|}\right] = \text{C/m}$, and hence $[\psi] = [r\phi] = \text{C}$.

On the other hand, the topological discussion is obscured for cylinder, since the field line potential ψ expressed by Eq. (5) is available only in a simply connected region [Fig. 1 (b)]. Even if we numerically extrapolate the electric field lines inside the cylinder, their topology will be the same as those created by a charged sphere, and therefore ψ is still expected to be single-valued in the right half plane ($r \in \mathbb{R}_{>0}, z \in \mathbb{R}$).

The value of the potential on the surface $r = R$ is of special interest. It is calculated via $I_{\text{hyg}}(m, \sqrt{1-m}; \pi)$, which corresponds to the boundary $|x| + |y| = 1$ where the series (2) converges.[1] It may be rewritten in several forms:

$$I_{\text{hyg}}(m, \sqrt{1-m}; \pi) = \pi\sqrt{1-m}\,F_2^{\text{Appell}}\left(\begin{smallmatrix}\frac{1}{2};1,\frac{1}{2}\\ 1,\frac{3}{2}\end{smallmatrix}; m, 1-m\right)$$
$$= \int_m^1 \frac{K(m)dm}{m\sqrt{1-m}} = \left[-\frac{\pi\mu}{8}F_3^4\left(\begin{smallmatrix}1,1,\frac{3}{2},\frac{3}{2}\\ 2,2,2\end{smallmatrix}; \mu\right) - \frac{\pi}{2}\ln\frac{-\mu}{16}\right]_{\mu=m/(m-1)}, \tag{26}$$

where $F_q^p\left(\begin{smallmatrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{smallmatrix}; x\right) = \sum_{j=0}^{\infty}\frac{(a_1)_j\dots(a_p)_j}{(b_1)_j\dots(b_q)_j}\frac{x^j}{j!}$ is the Barnes (generalized) hypergeometric function.

Derivation of Eq. (3) — Finally, we derive Eq. (3). The naive expansion gives

$$I_{\text{hyg}} = 2As\sum_{l,j,k=0}^{\infty}\frac{(1)_j(\frac{1}{2})_{l+j}(\frac{1}{2})_k(\frac{1}{2})_{l+k}}{(\frac{3}{2})_j(\frac{3}{2})_{l+k}}\frac{(ms^2)^l(A^2)^j(s^2)^k}{l!\,j!\,k!}. \tag{27}$$

This triple sum does not reduce to the Lauricella function;[2] we need a three-variable analog of the Kampé de Fériet function for it. Performing two of three summations, we obtain three different expressions:

$$I_{\text{hyg}} = 2As\sum_{l=0}^{\infty}\frac{(\frac{1}{2})_l}{2l+1}\frac{(ms^2)^l}{l!}F_1^2\left(\begin{smallmatrix}1,\frac{1}{2}+l\\ \frac{3}{2}\end{smallmatrix}; A^2\right)F_1^2\left(\begin{smallmatrix}\frac{1}{2},\frac{1}{2}+l\\ \frac{3}{2}+l\end{smallmatrix}; s^2\right) \tag{28}$$

$$= 2As\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j}{(\frac{3}{2})_j}(A^2)^j F_1^{\text{Appell}}\left(\begin{smallmatrix}\frac{1}{2};\frac{1}{2}+j,\frac{1}{2}\\ \frac{3}{2}\end{smallmatrix}; ms^2, s^2\right) \tag{29}$$

$$= 2As\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{(\frac{3}{2})_k}\frac{(s^2)^k}{k!}F_2^{\text{Appell}}\left(\begin{smallmatrix}\frac{1}{2};\frac{1}{2}+k,1\\ \frac{3}{2}+k,\frac{3}{2}\end{smallmatrix}; ms^2, A^2\right), \tag{30}$$

where $F_1^2\left(\begin{smallmatrix}\alpha,\beta\\ \gamma\end{smallmatrix}; x\right)$ and $F_1^{\text{Appell}}\left(\begin{smallmatrix}\alpha;\beta,\beta'\\ \gamma\end{smallmatrix}; x, y\right)$ are Gauss's and Appell's hypergeometric function, respectively. None of them, however, can be used to obtain a summation-symbol-free expression for definite integral $s = 1$. To this end, we apply the linear transformation 15.3.6 in Ref. 13 to the rightmost F_1^2 function in Eq. (28). Performing re-summation after this rewriting, we get Eq. (3).

Integration of the product of Jacobi zeta and elliptic function — We further have a by-product formula as a bonus. Recalling another form of the elliptic integral of the third kind Π in terms of the Jacobi zeta and theta functions,[11] Eq. (3) is re-interpreted as

$$\int_0^u Z(u|m)\,\text{sc}(u|m)du = -\,\text{am}(u|m)$$
$$+\frac{\pi\,\text{sc}(u|m)}{2K(m)}F_2^{\text{Appell}}\left(\begin{smallmatrix}\frac{1}{2};\frac{1}{2},1\\ 1,\frac{3}{2}\end{smallmatrix}; m, (m-1)\,\text{sc}^2(u|m)\right). \tag{31}$$

Here, we comment on the significance of the formula (31). Let JE denote any of the twelve Jacobi elliptic functions. While the integral of $Z * \text{JE}^2$ is expressible by functions appearing in the classical theory of elliptic functions,[12,13] it is impossible for $Z * \text{JE}$. Therefore, providing formulas similar to Eq. (31) for all remaining Jacobi elliptic functions, it fills in the missing piece of the table of formulas.[15]

Conclusion — We have presented the closed-form expressions of electric potentials and field lines for a uniformly-charged cylinder and tube, expressed by elliptic integrals and Appell's hypergeometric functions. The field lines are plotted as equipotential lines of the field line potential (5), defined only in chargeless regions. The field line potential can have a multi-valued character if the chargeless region is not simply connected and the topological charge can be introduced. The integration formula (31) for the product of the Jacobi zeta and elliptic functions has also been presented, which was absent in classical tables of formulas.

The same problem for low-dimensional objects including disk and ring is soon solved as a corollary. Writing ψ in $\rho \neq 0$ region seems challenging. Finding another nice application of special functions for other geometrical shapes remains open.

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15) A similar example of the classically inexpressible integrals of elliptic functions written by F_1^{Appell} is the integral of JE^{α} with noninteger α. It has already been implemented by Mathematica.